TEL AVIV UNIVERSITY

Chasing Positive Bodies

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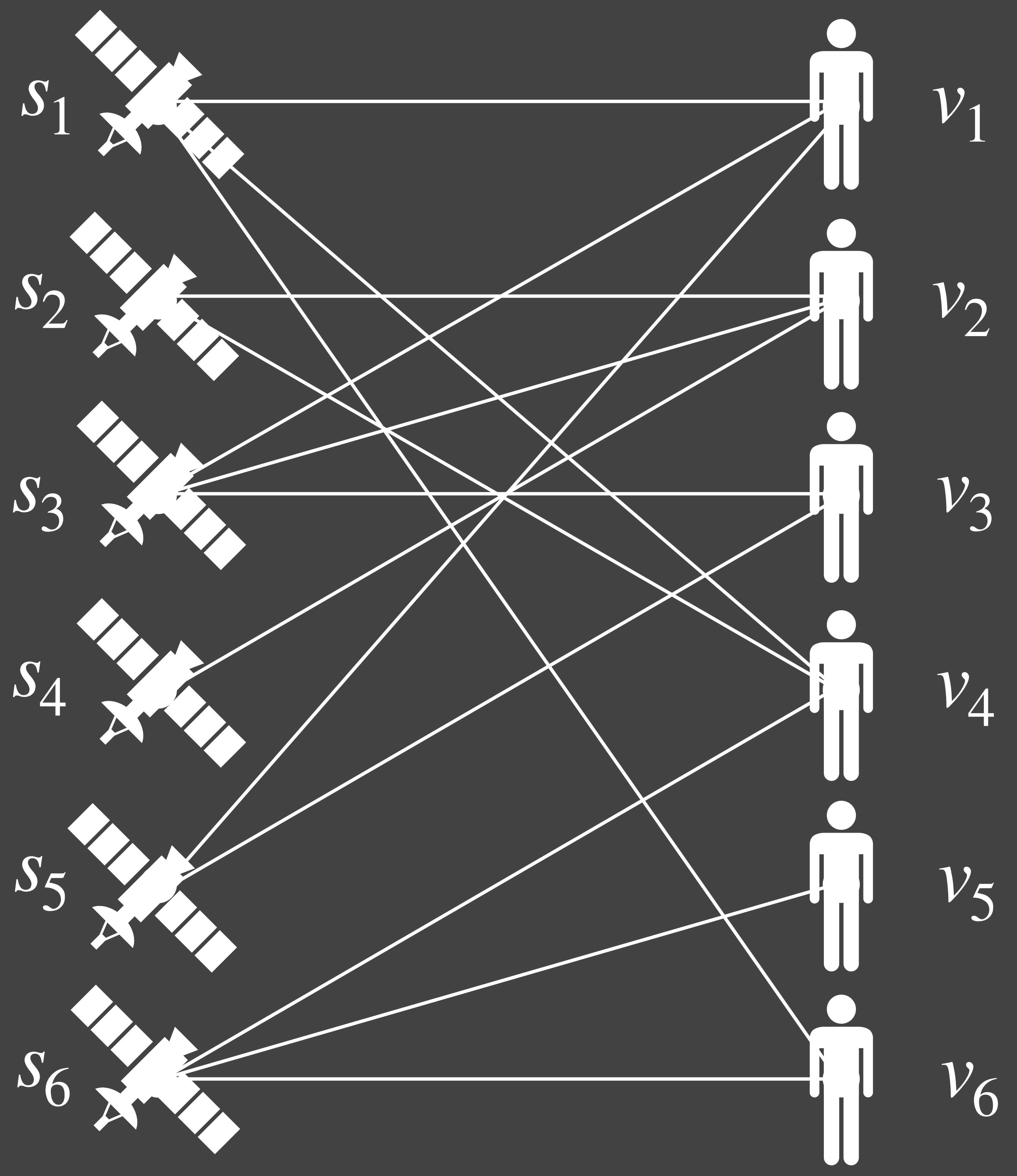
(U. of Warwick)

Niv Buchbinder
(Tel Aviv U.)

Thatchaphol Saranurak
(U. of Michigan)

Introduction

Motivating Problem



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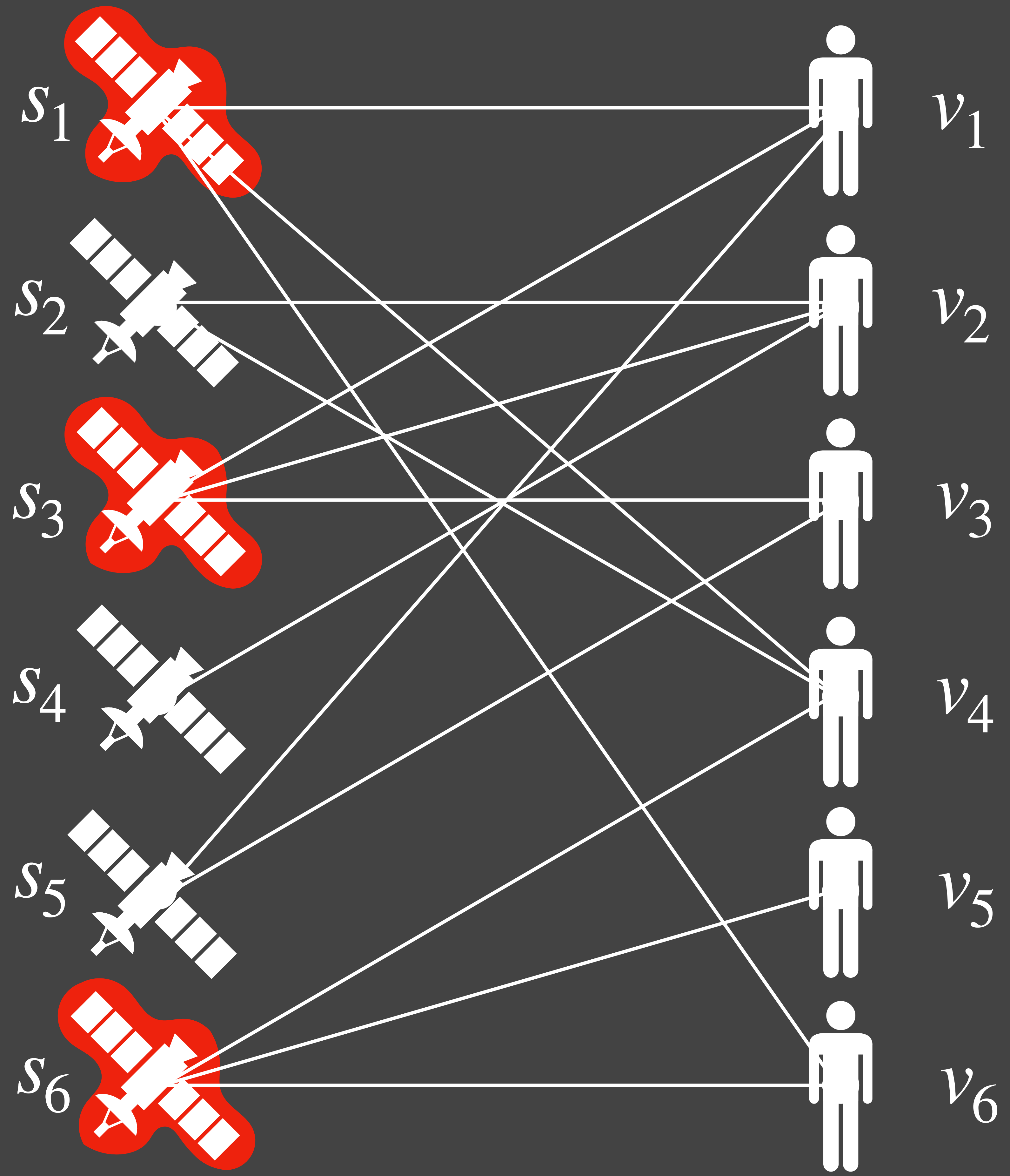

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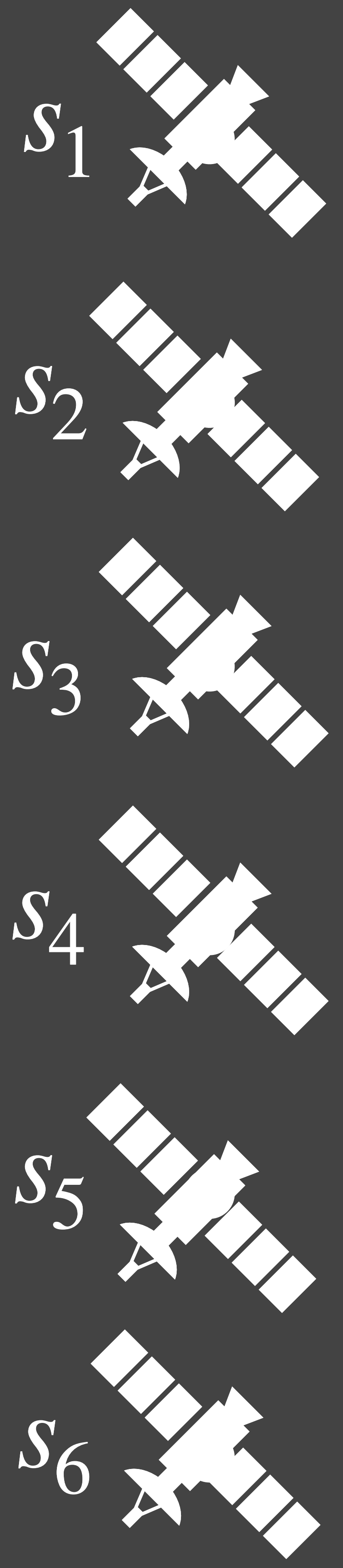

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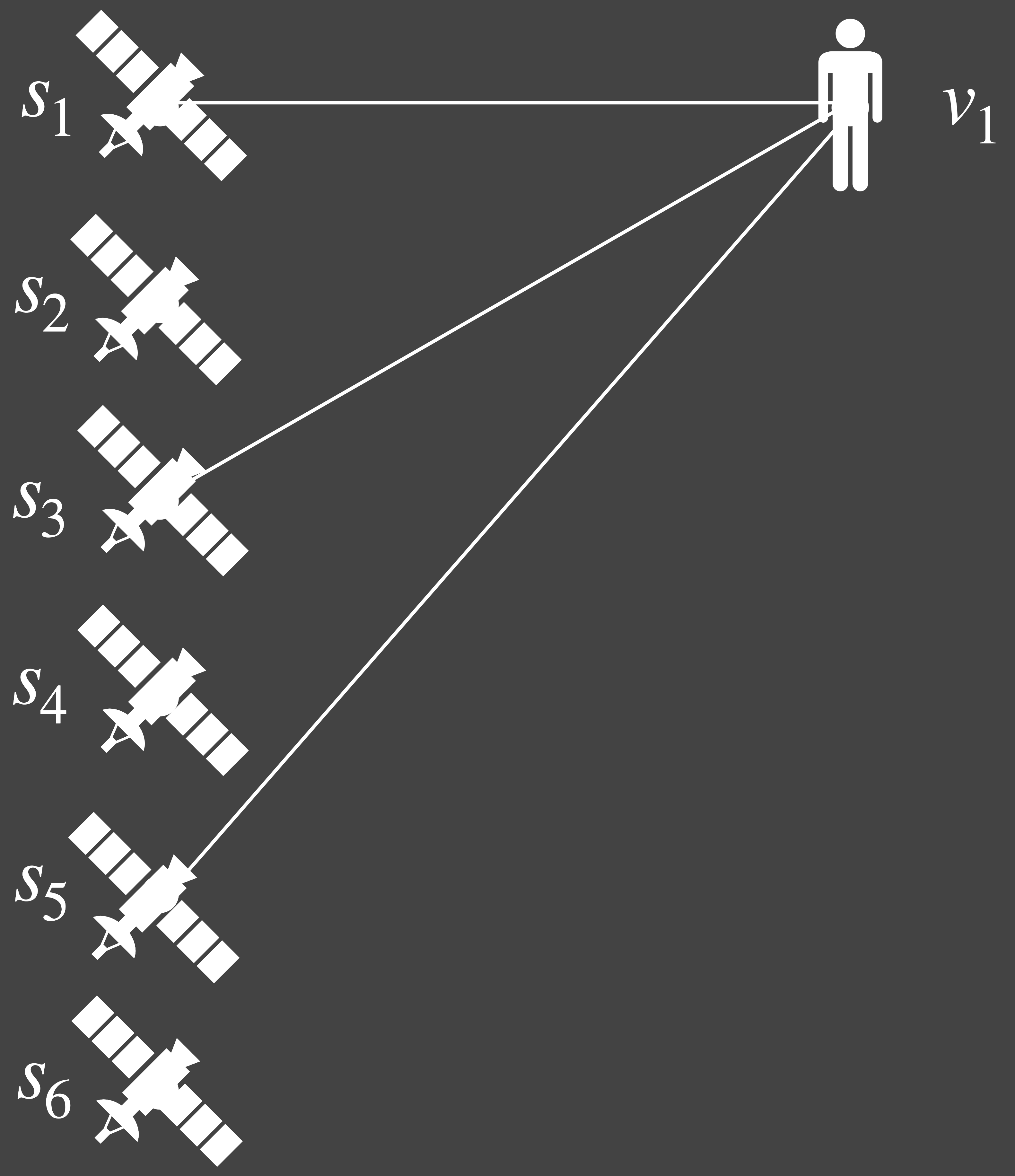

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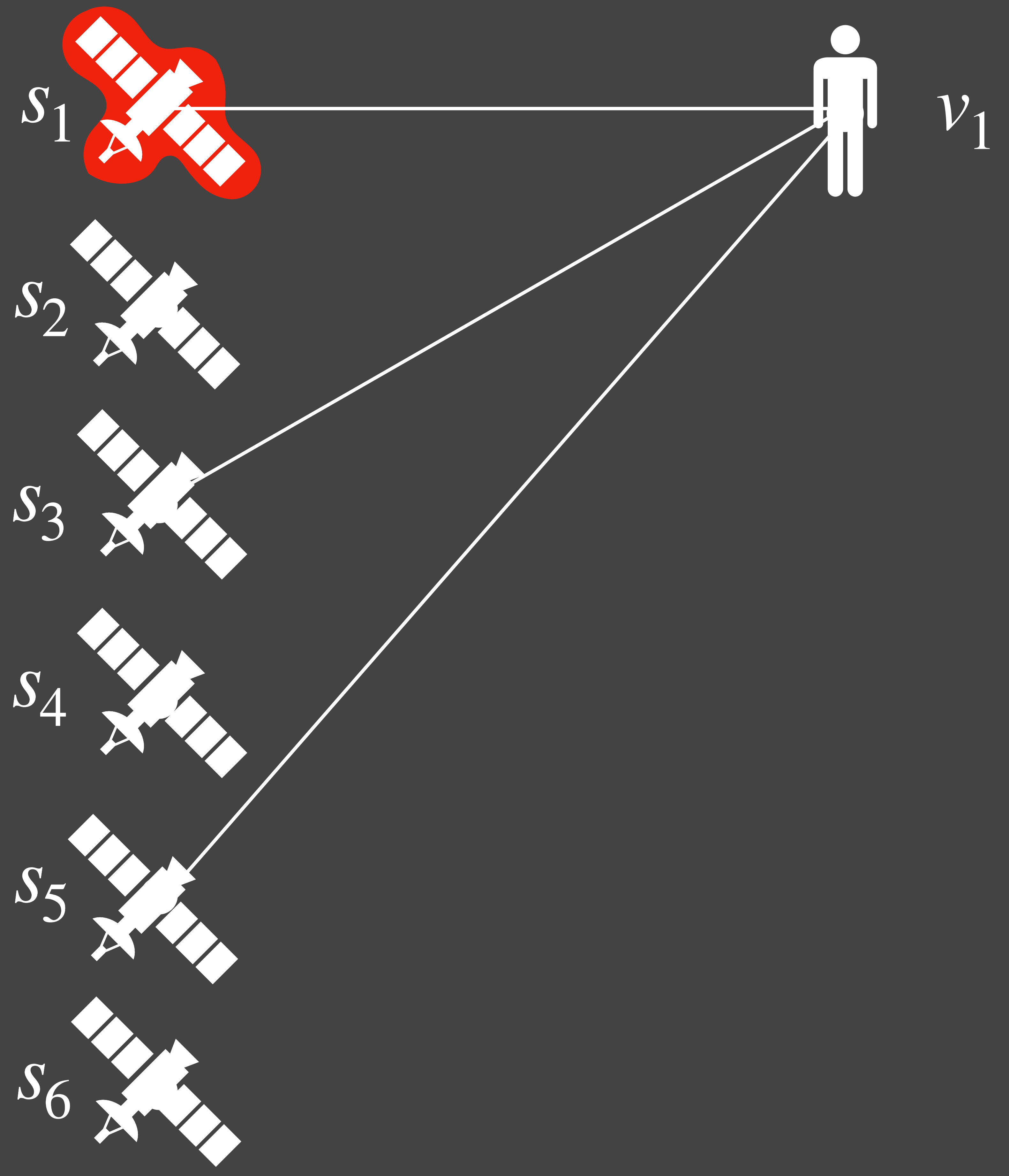

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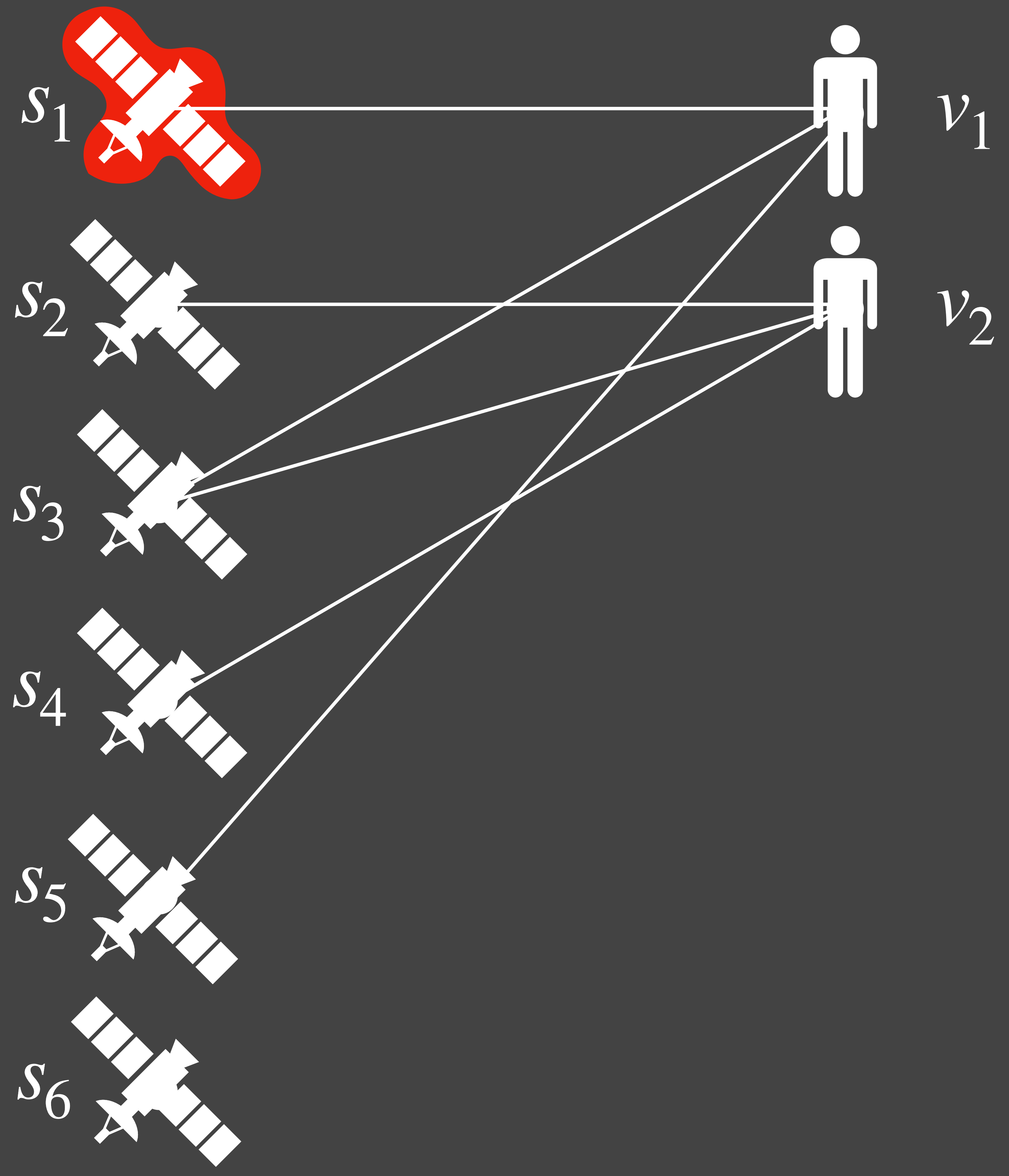

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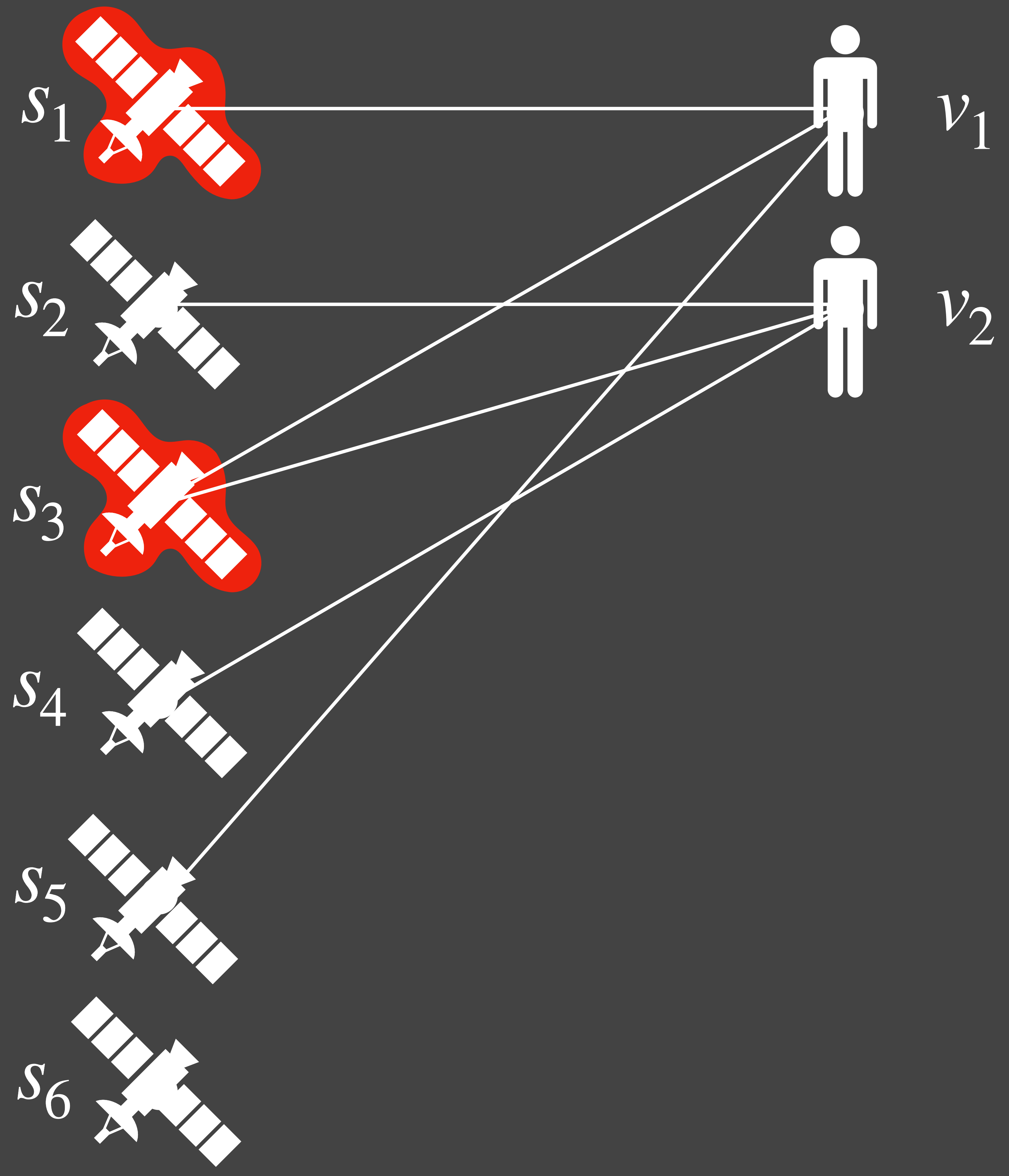

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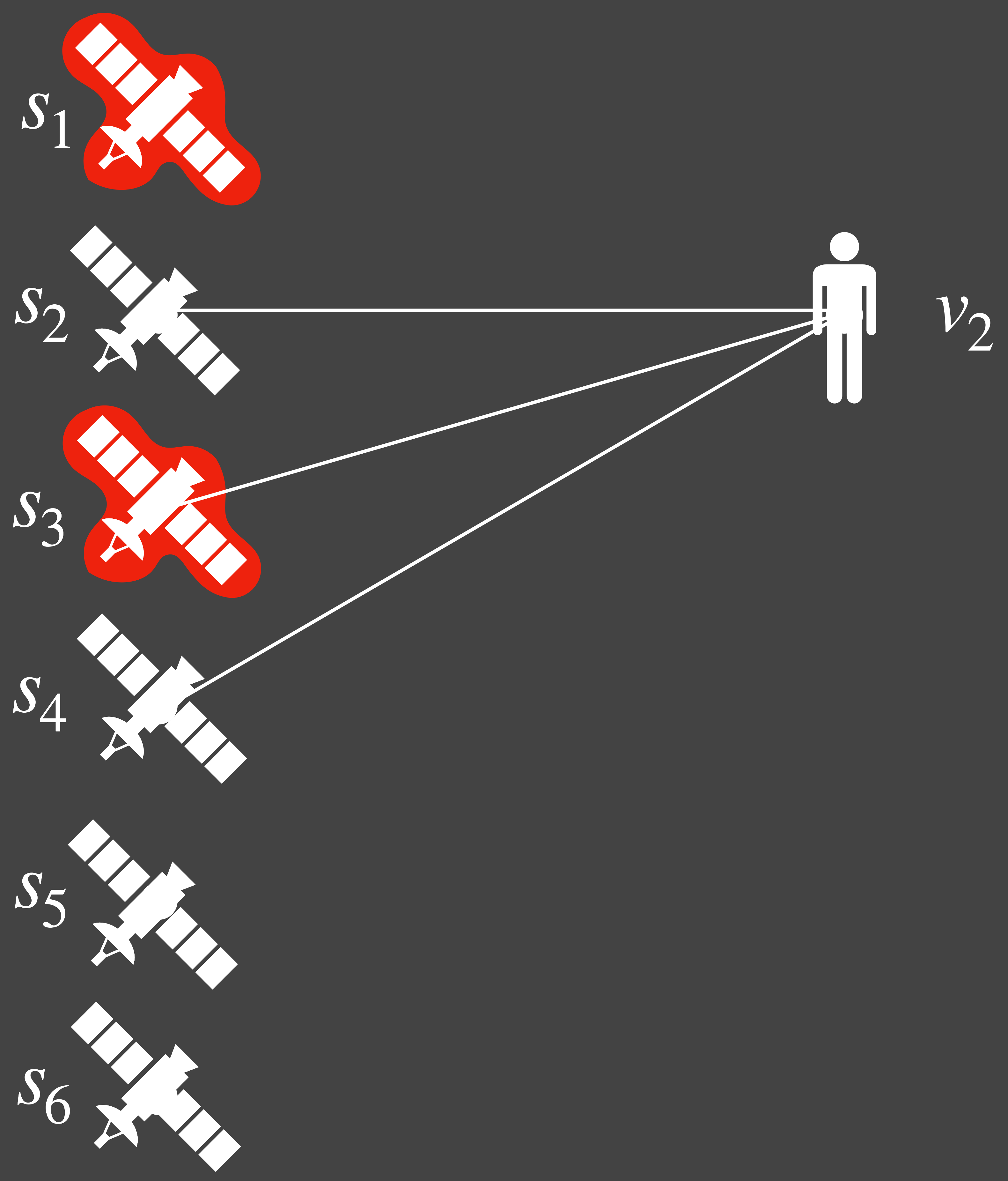

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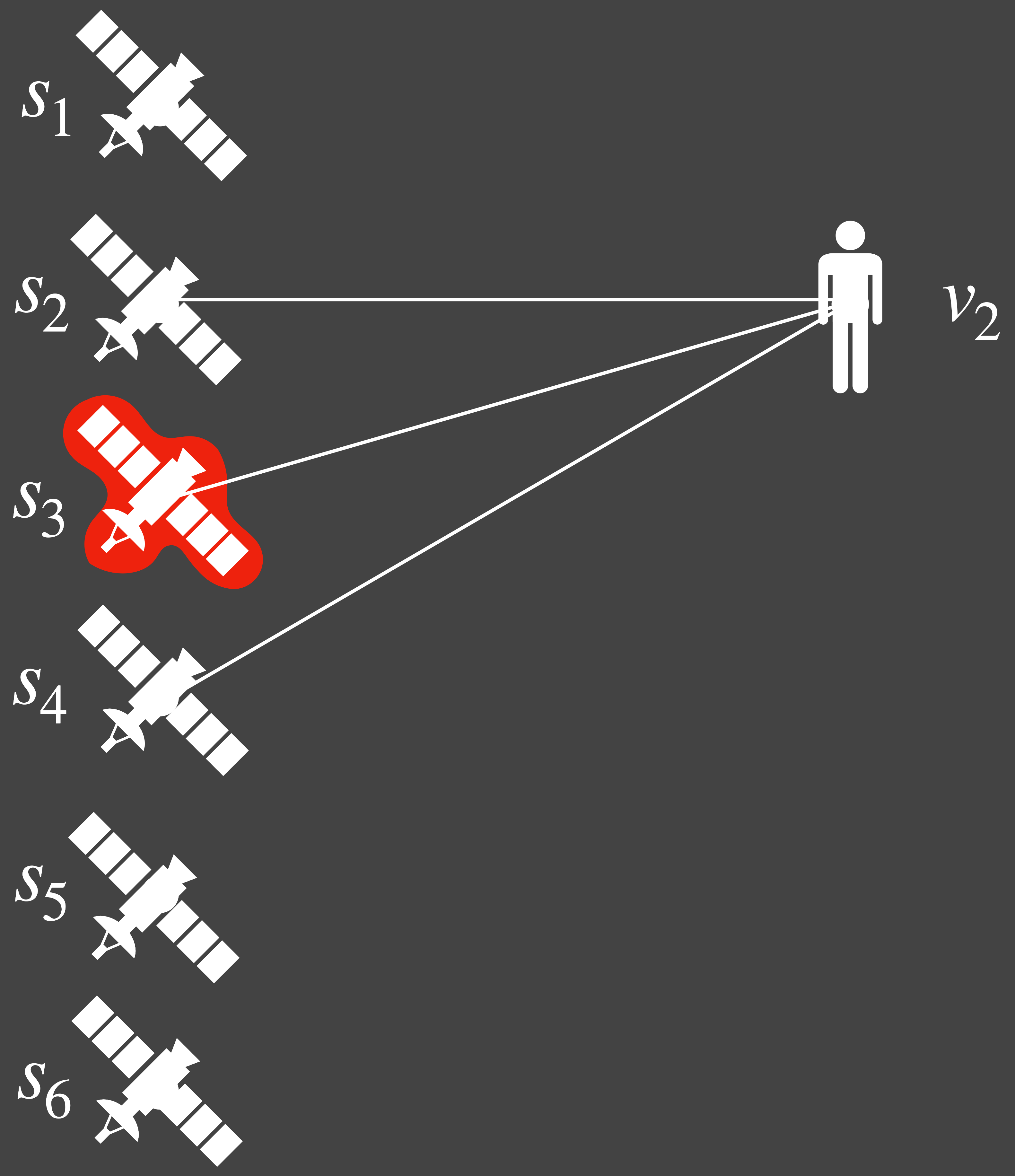

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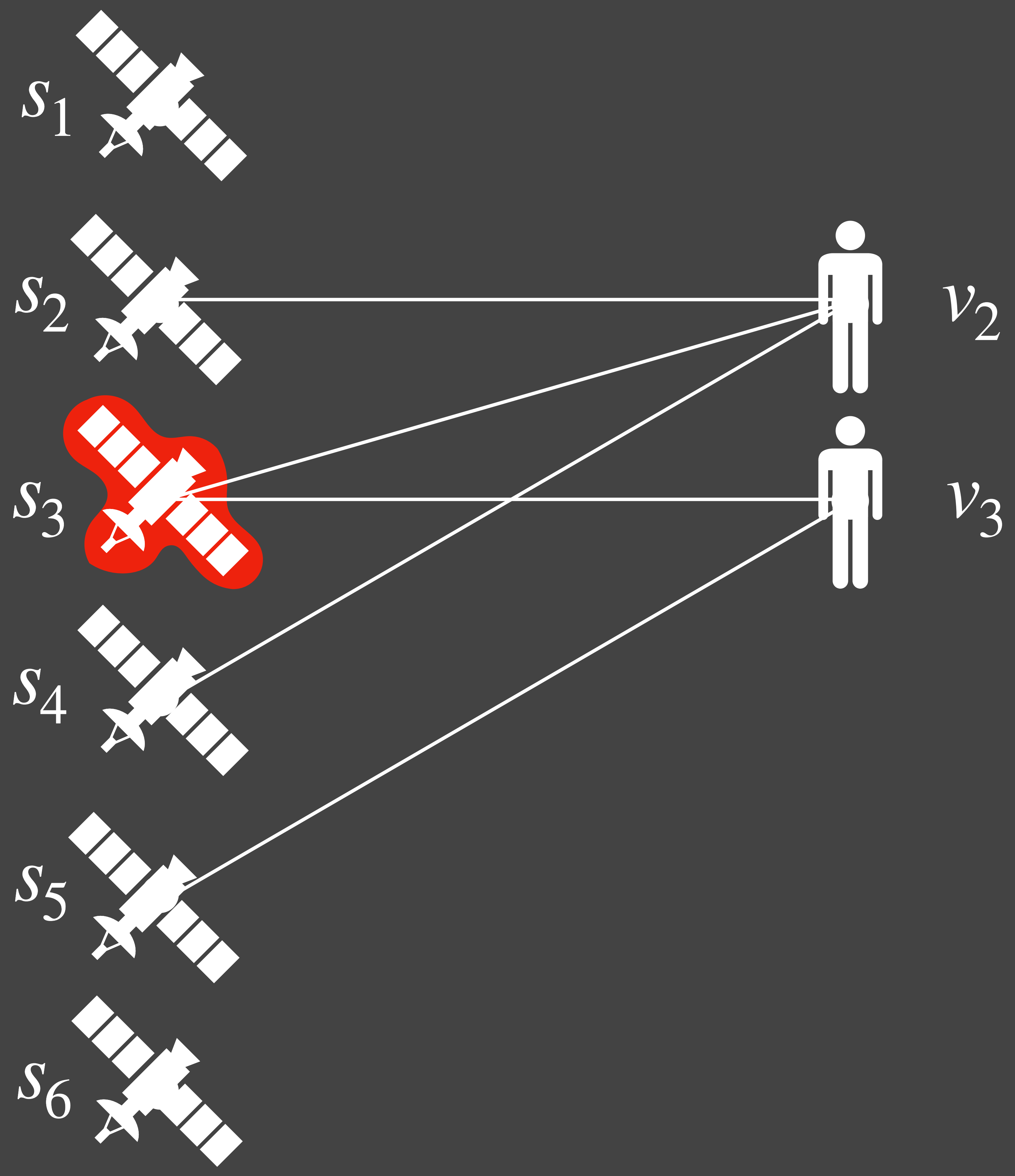

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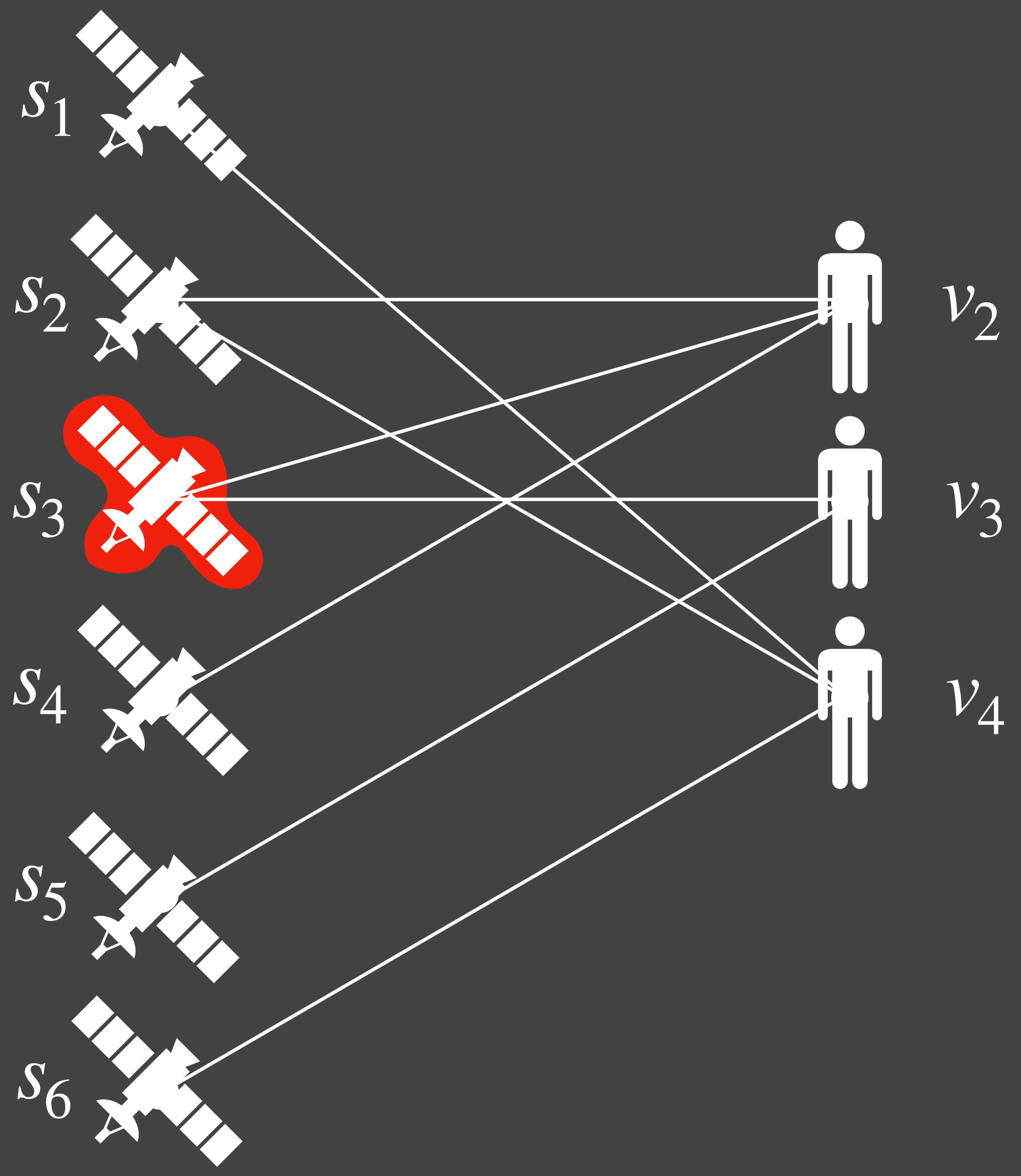

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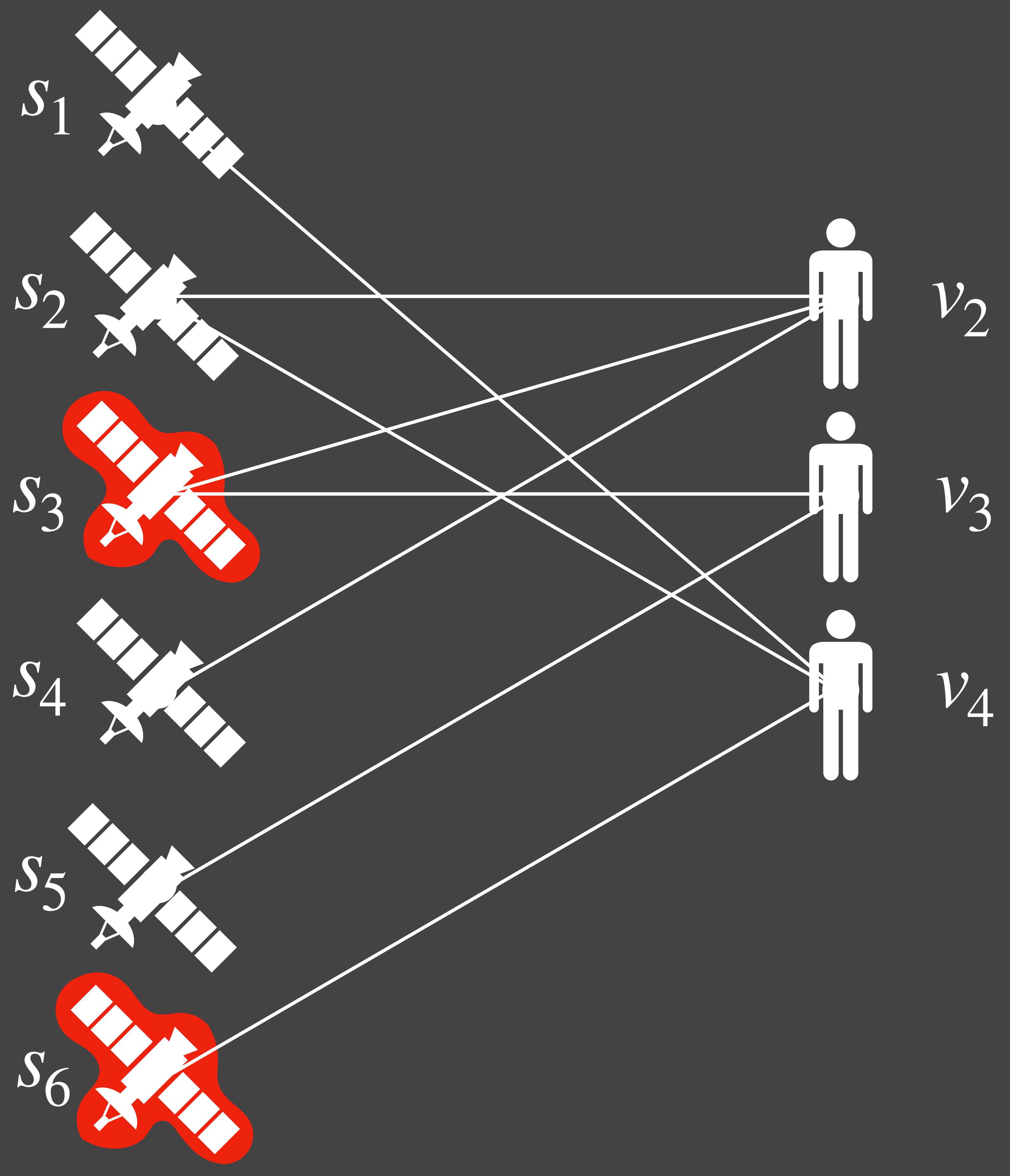

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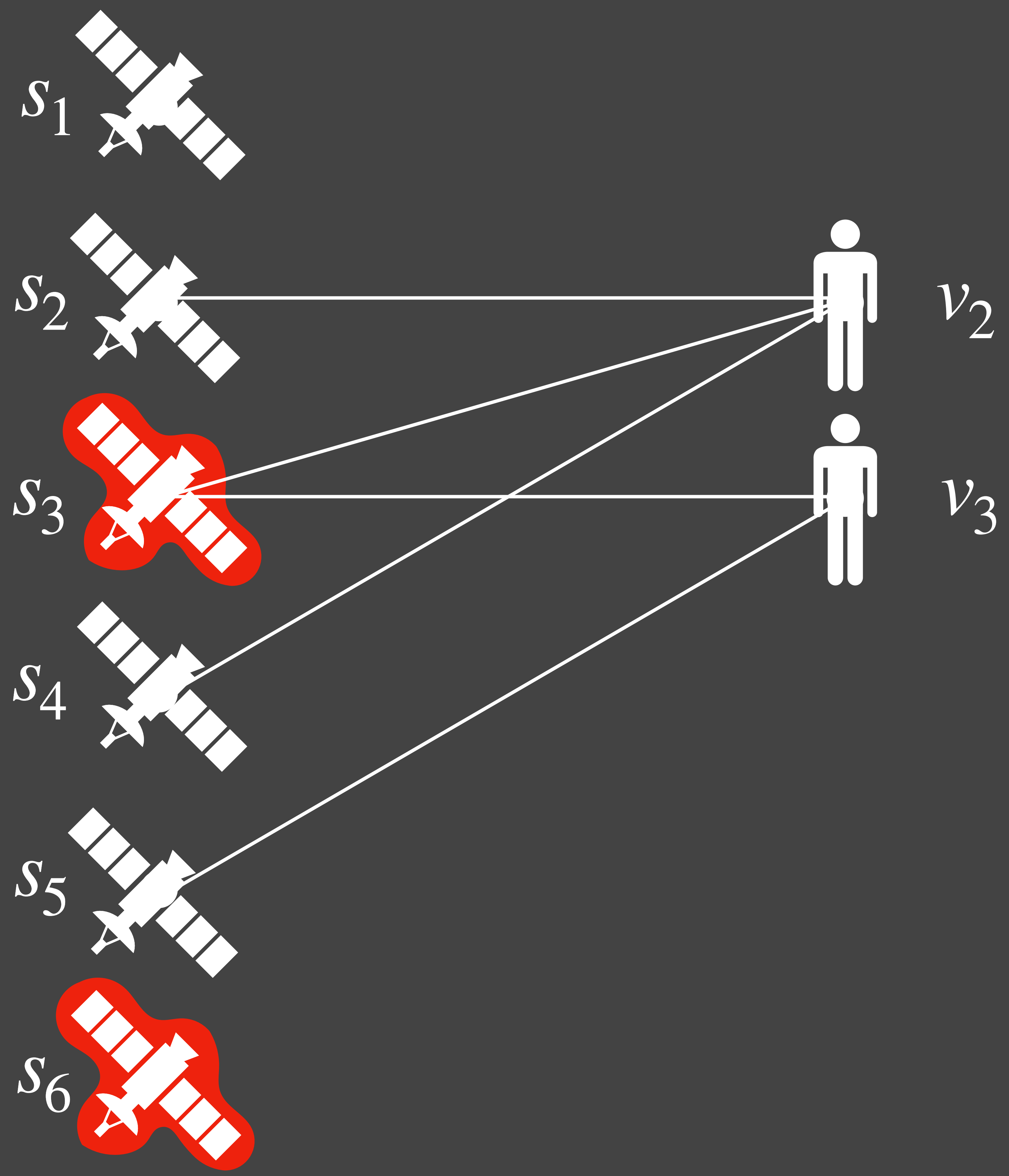

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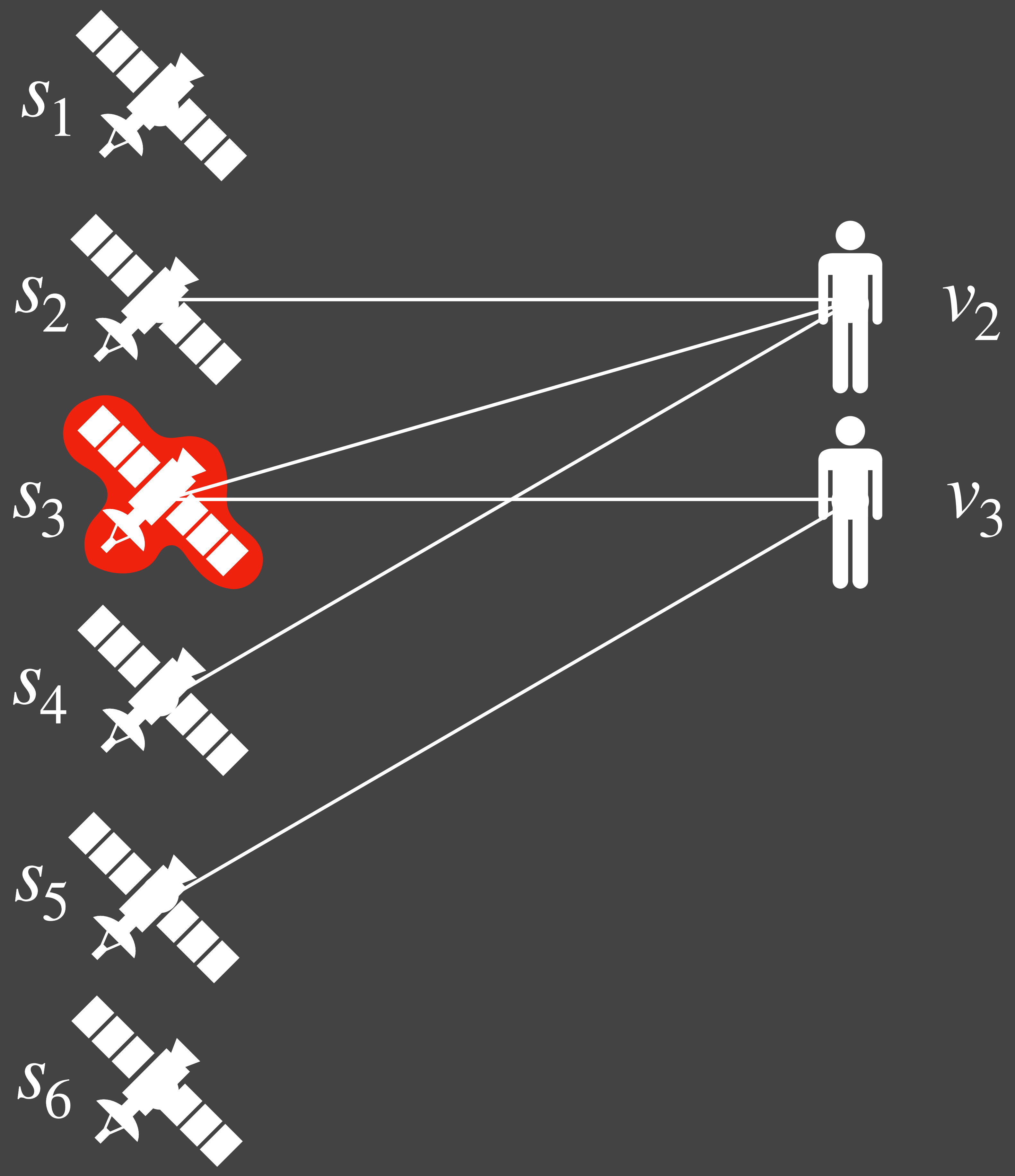

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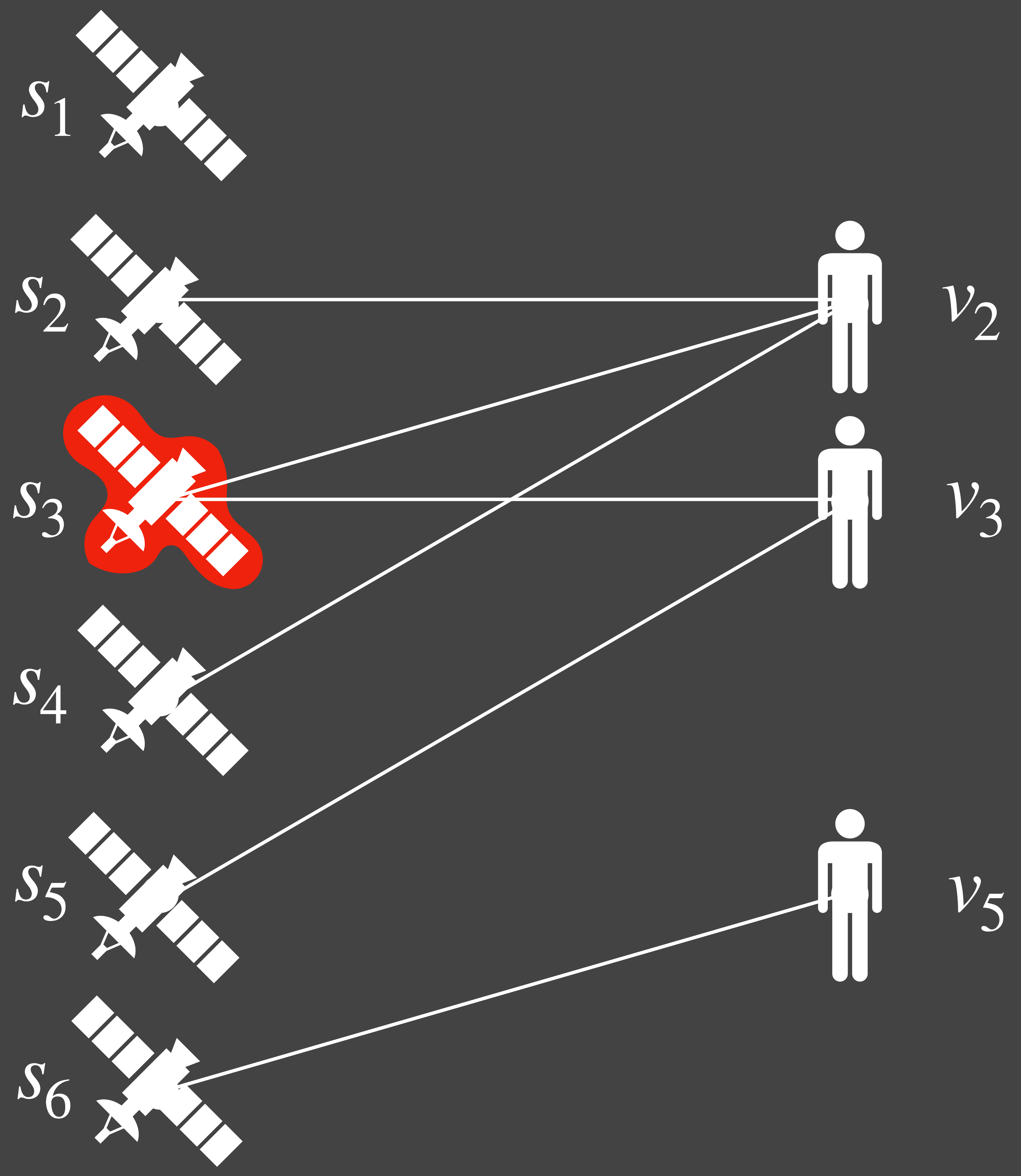

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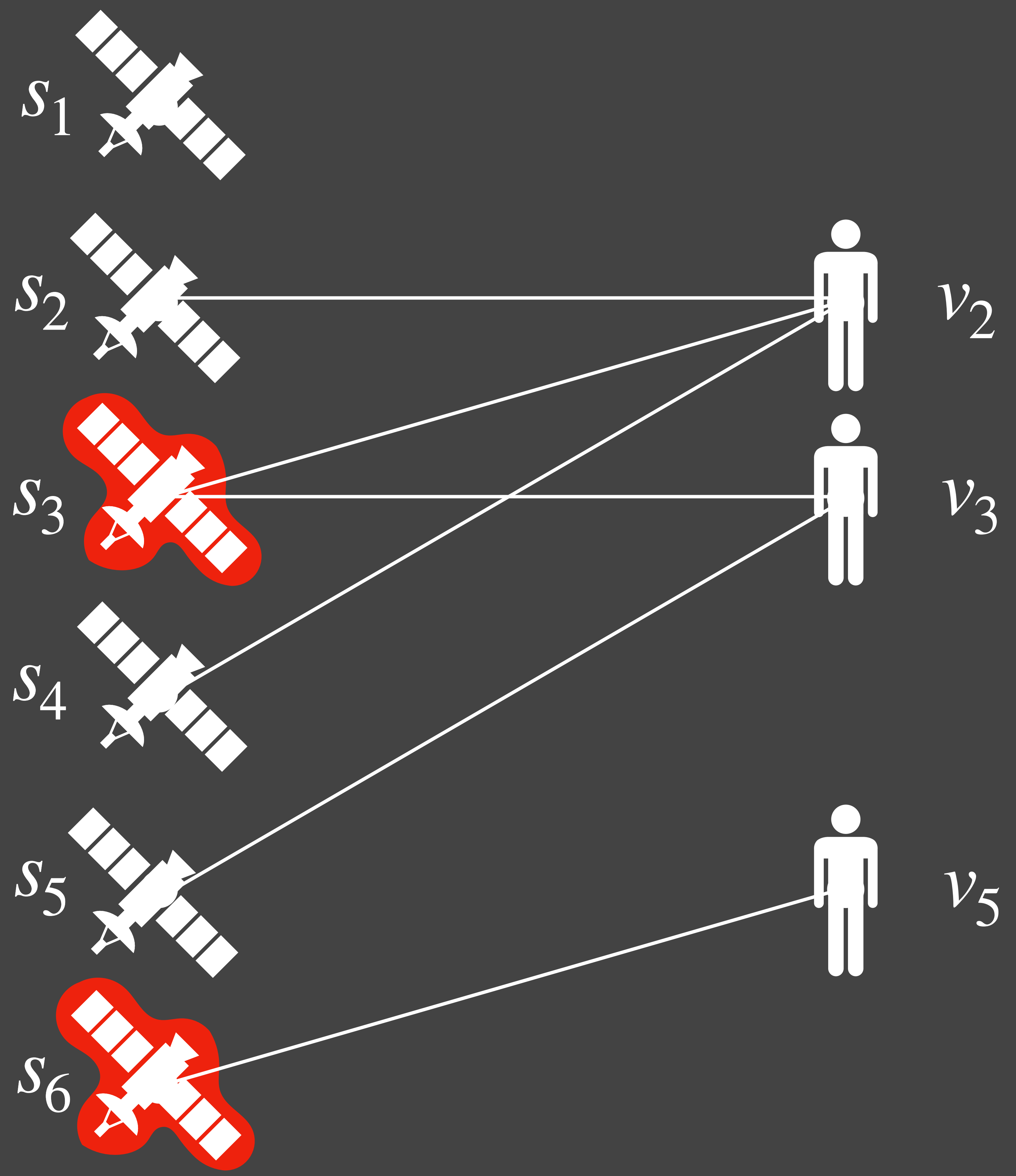

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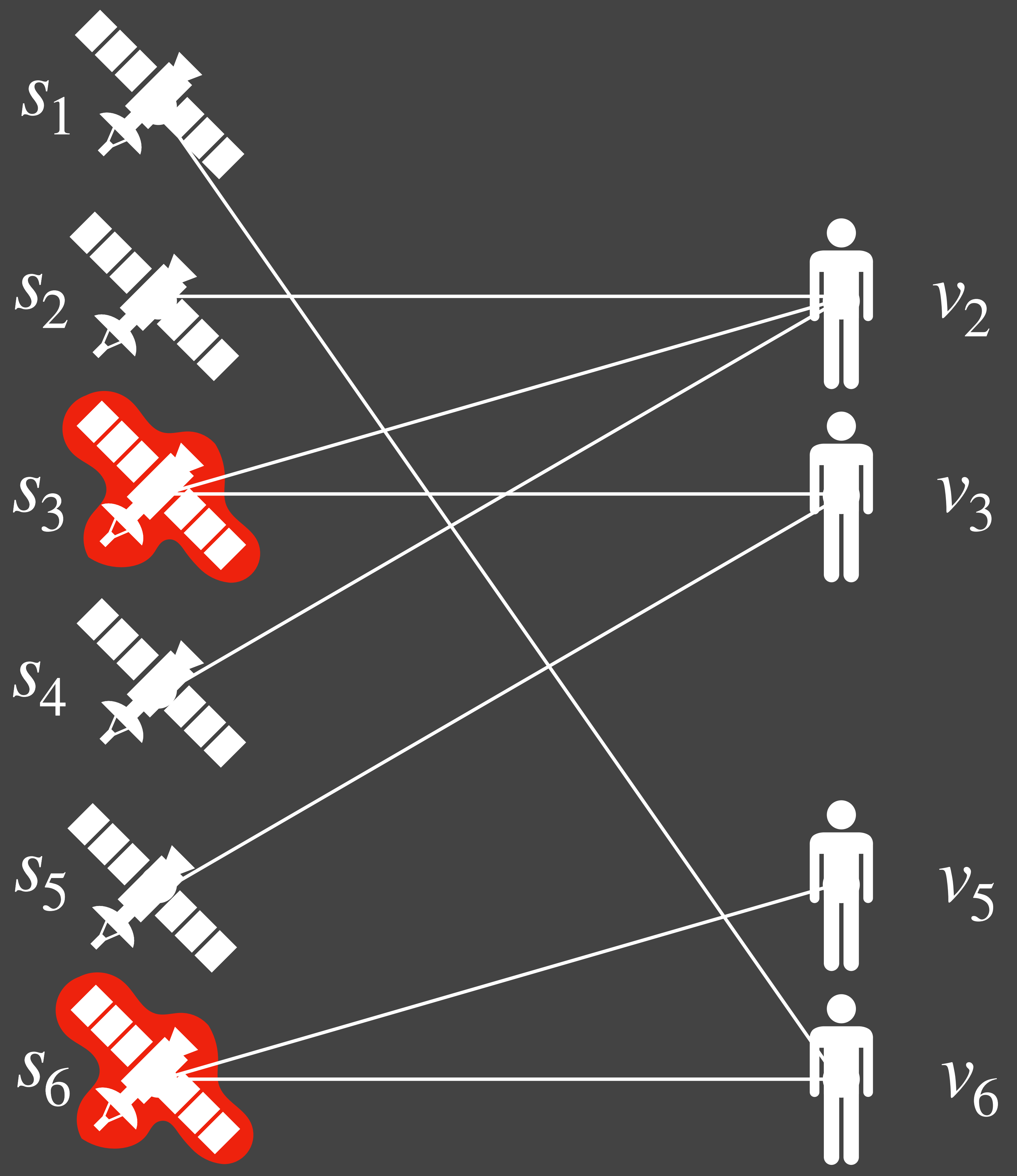

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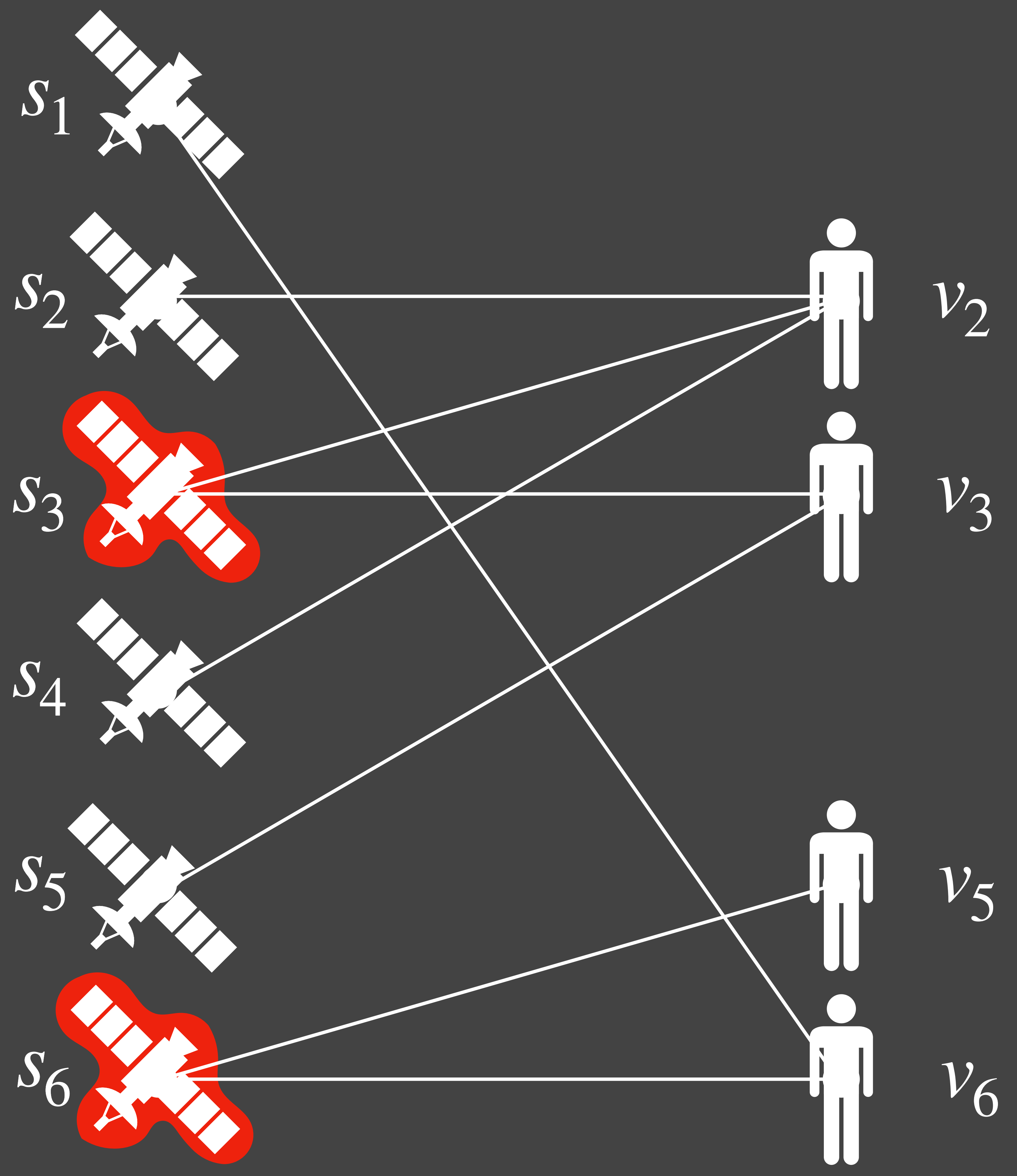

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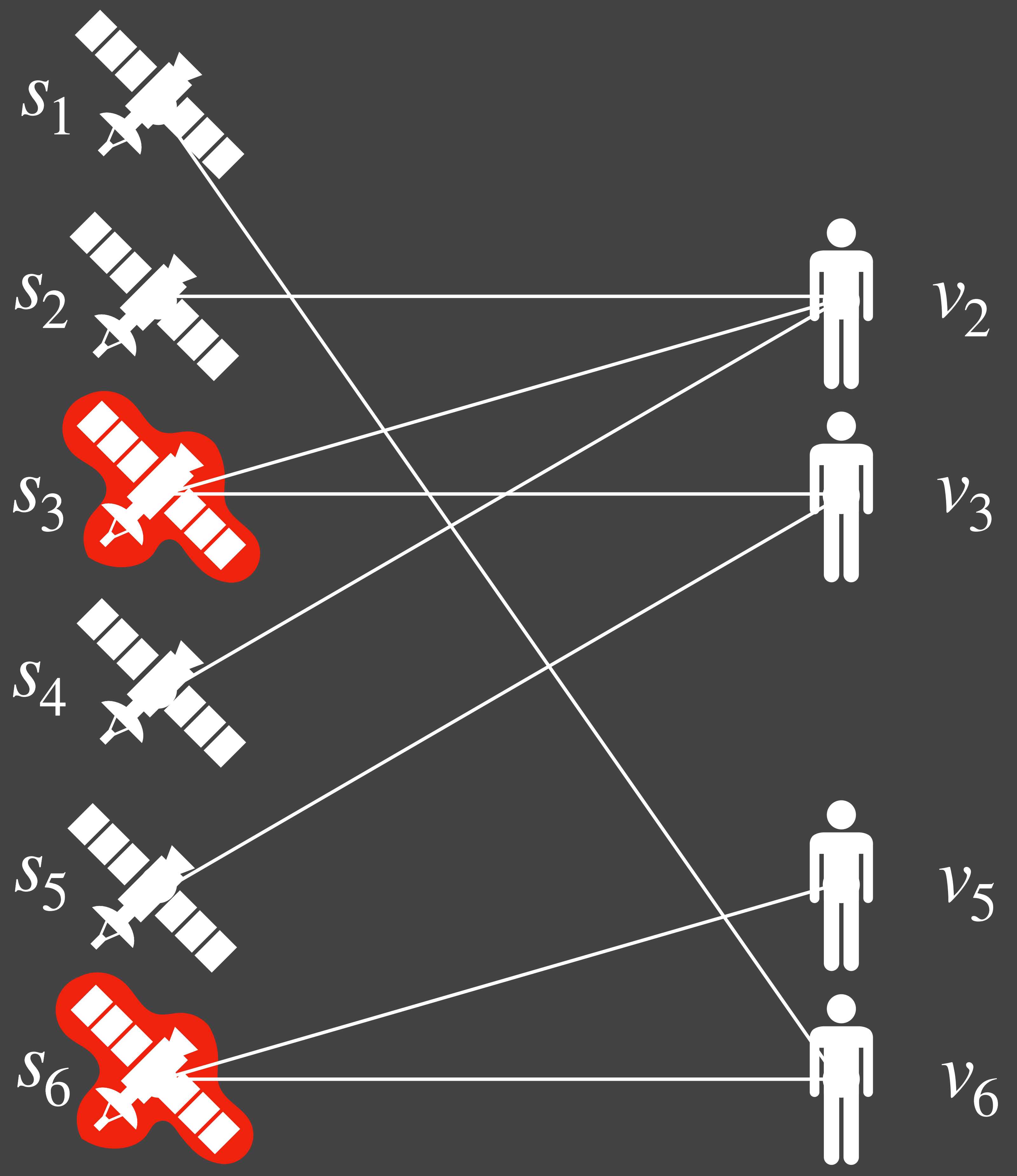

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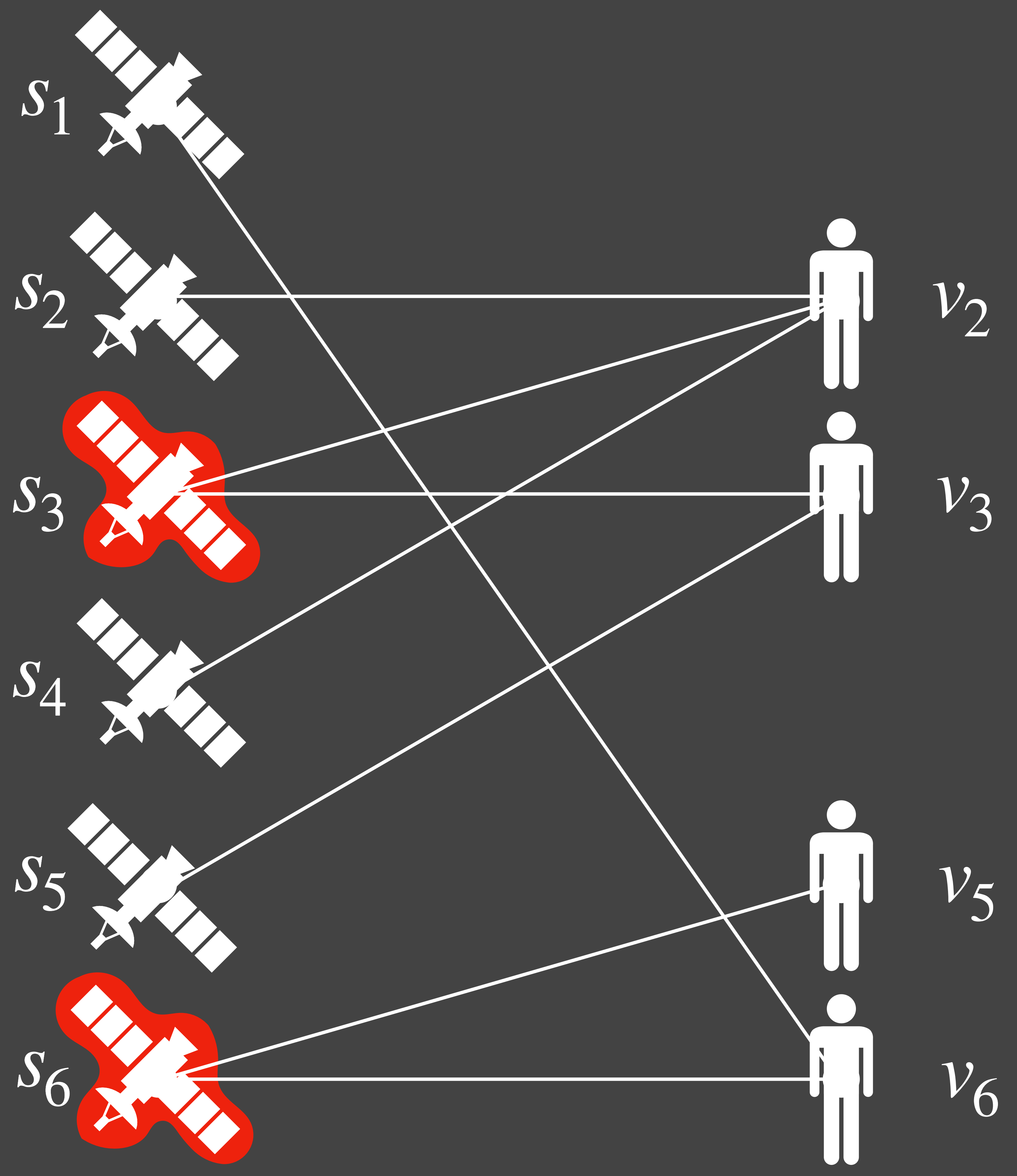

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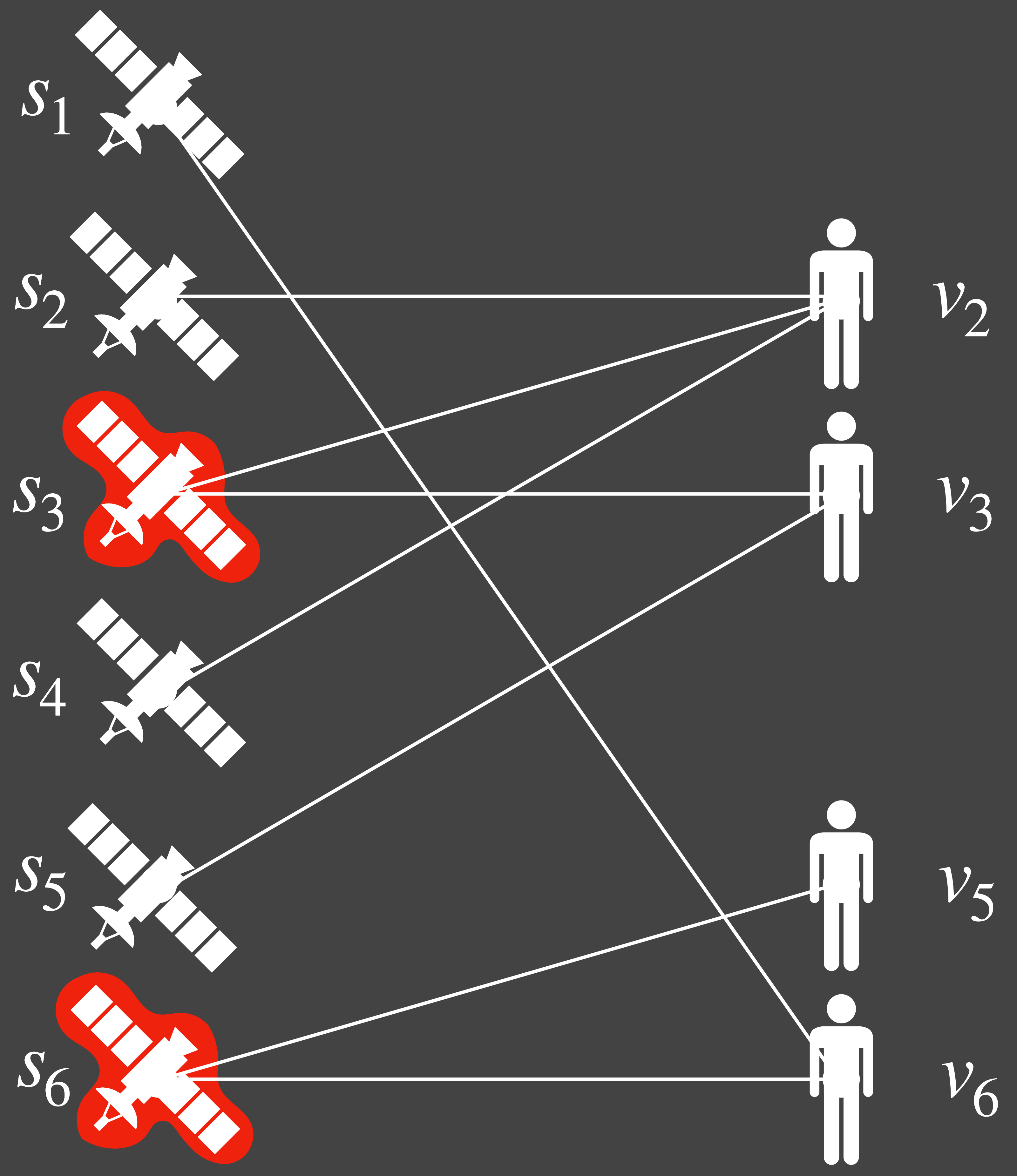

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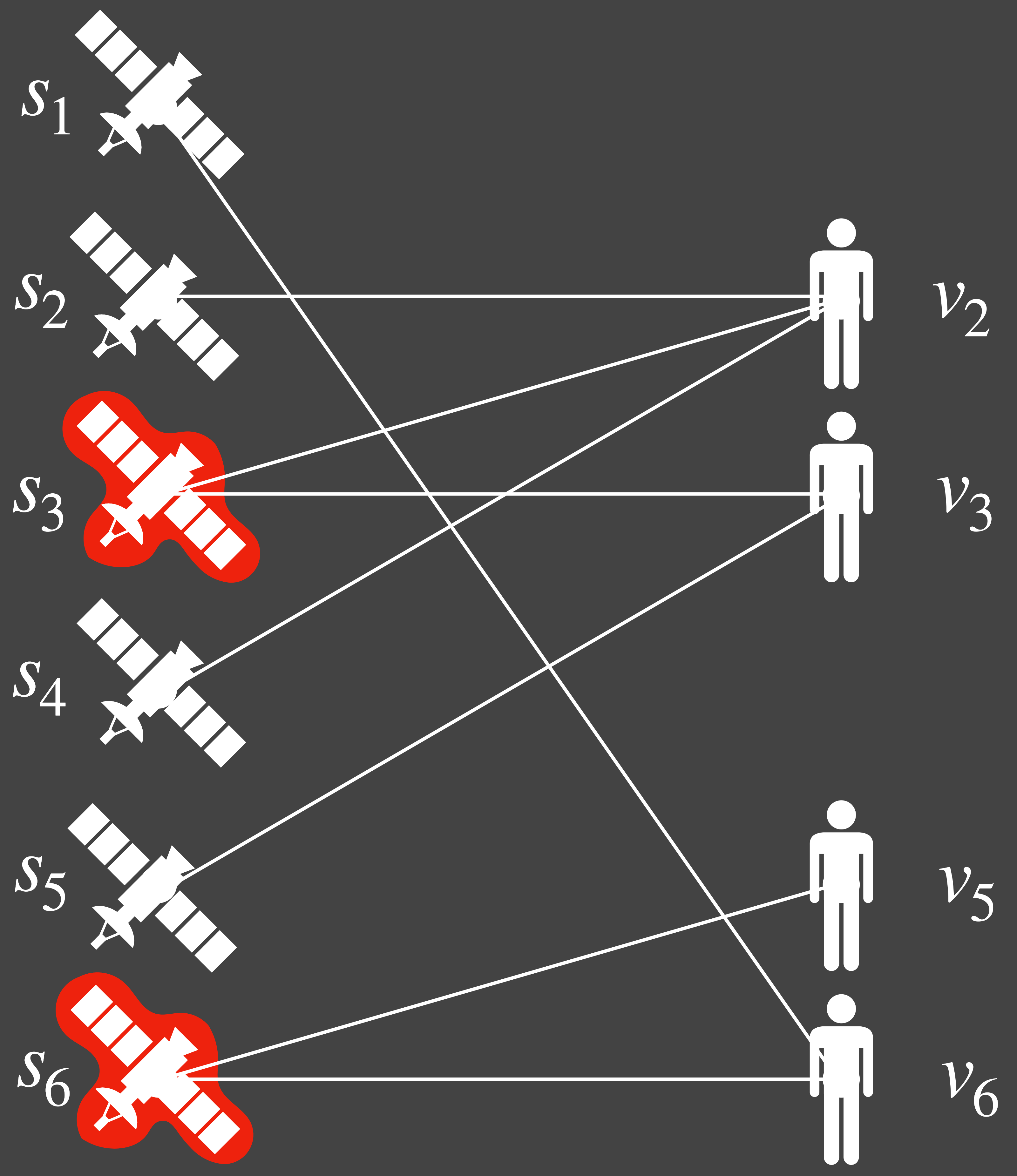

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Q: What is recourse/ approximation tradeoff?

Low Recourse Dynamic Algorithms

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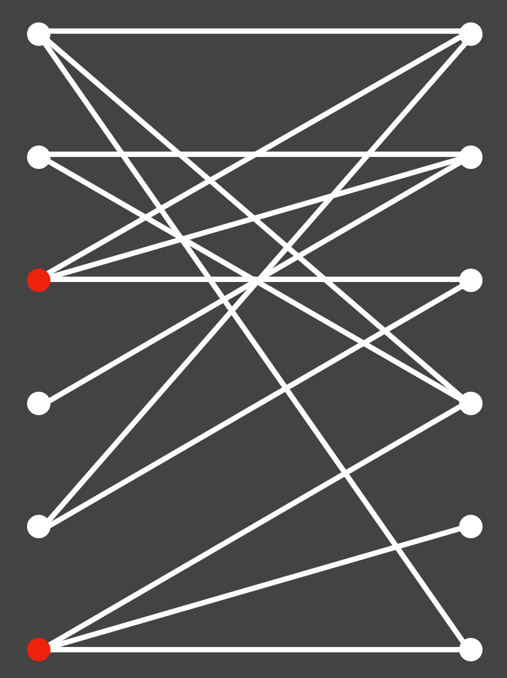

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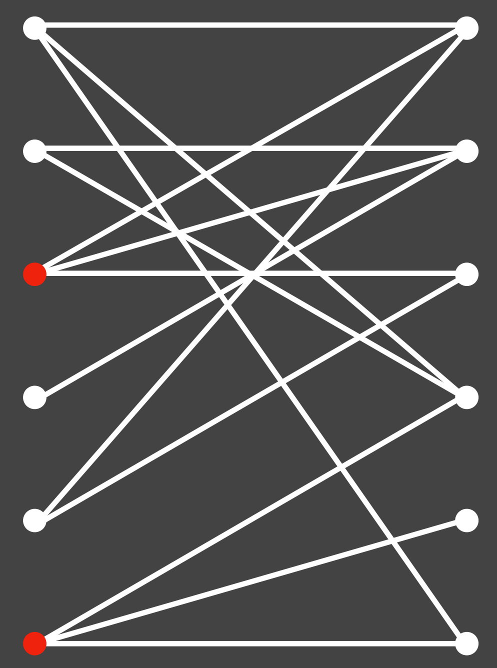

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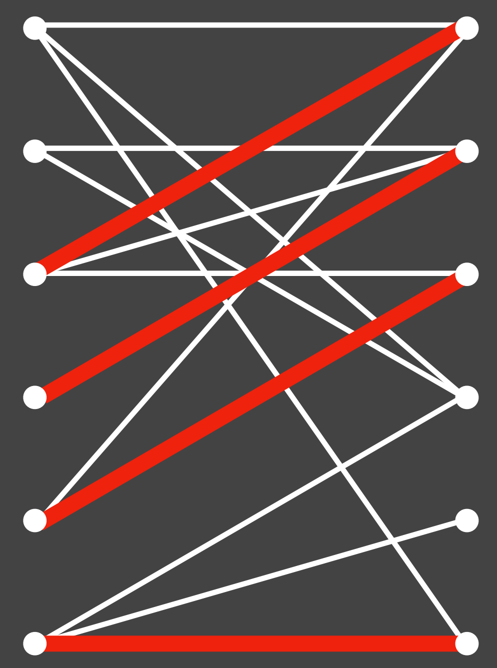

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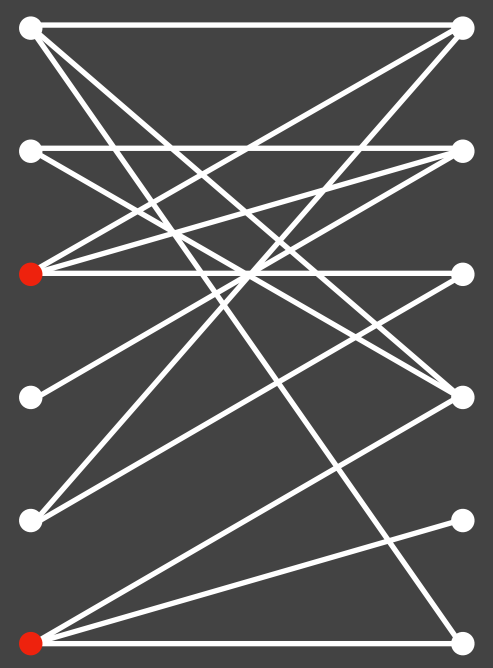

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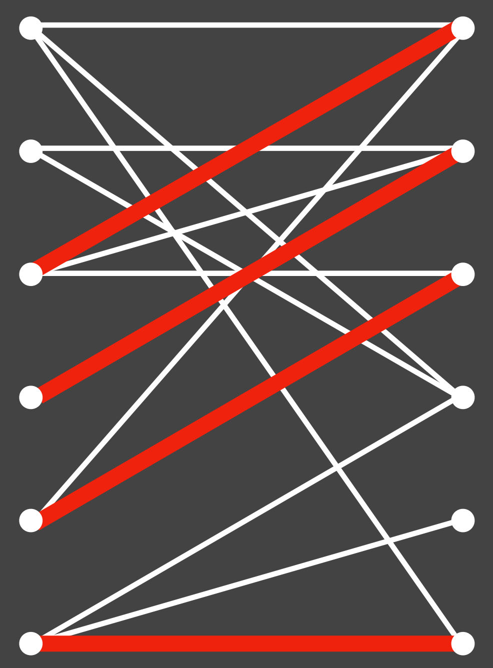

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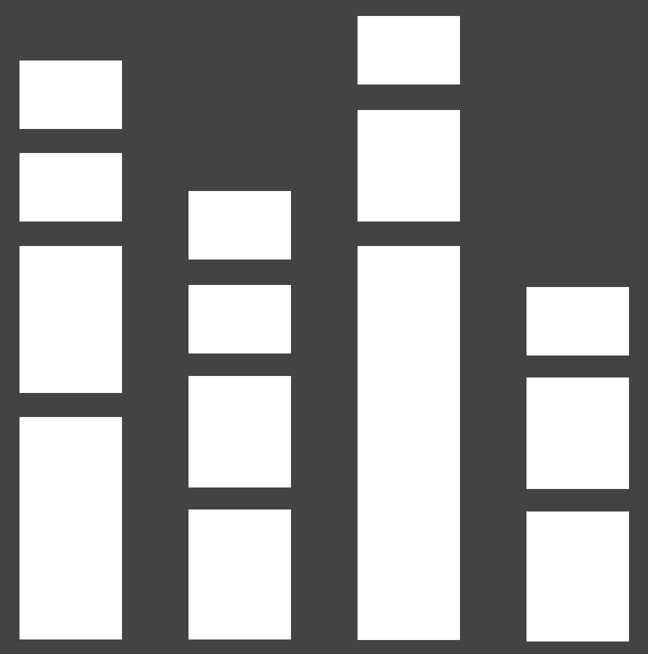

Load Balancing

Low Recourse Dynamic Algorithms

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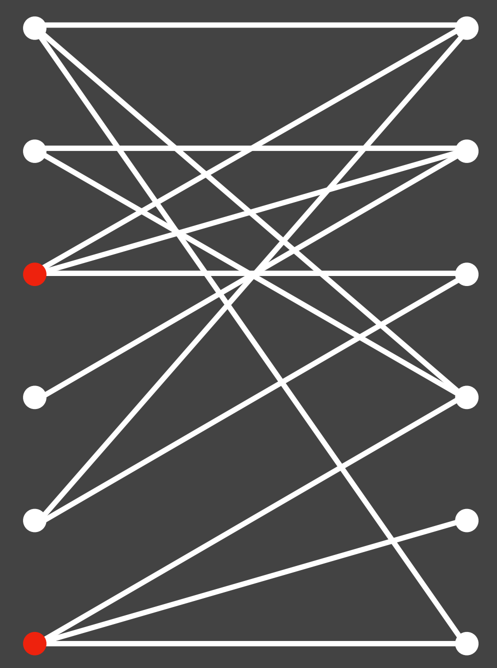

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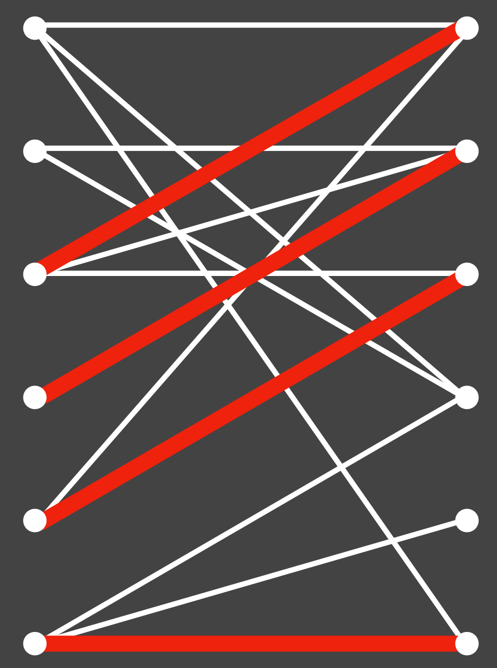

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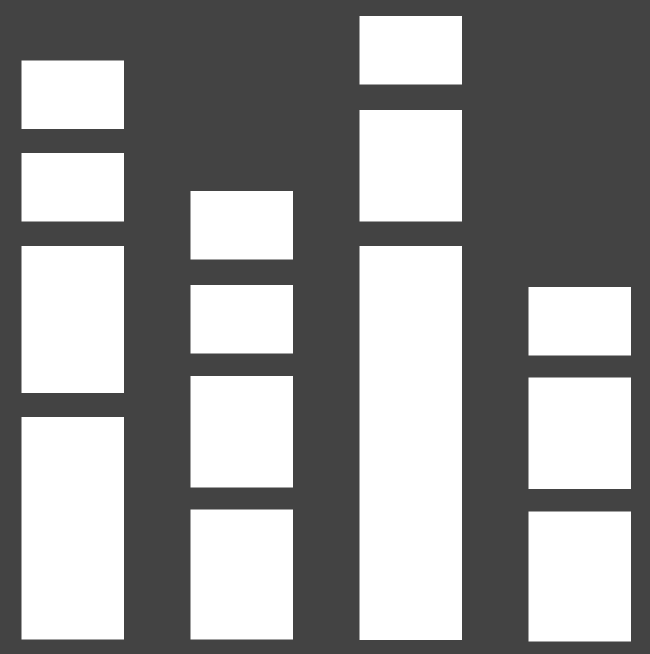

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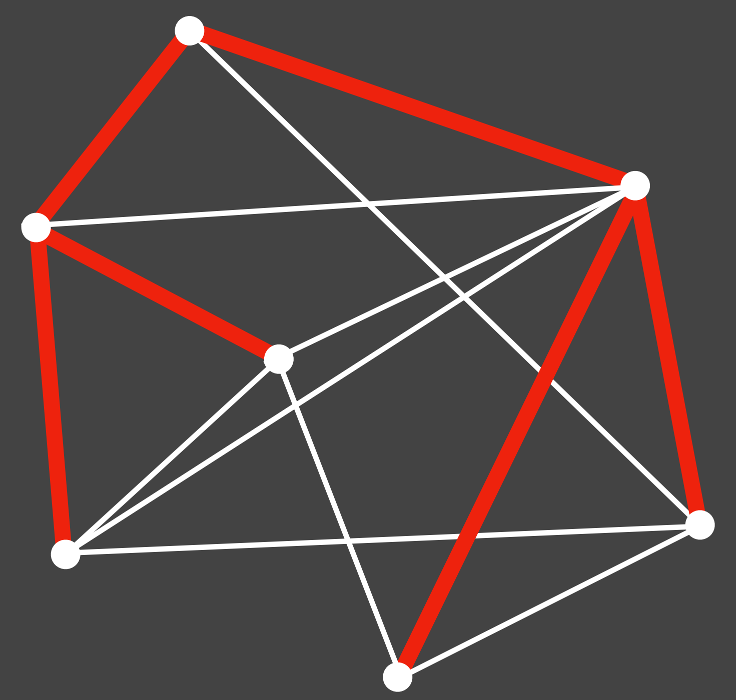

Minimum Spanning Tree

Low Recourse Dynamic Algorithms

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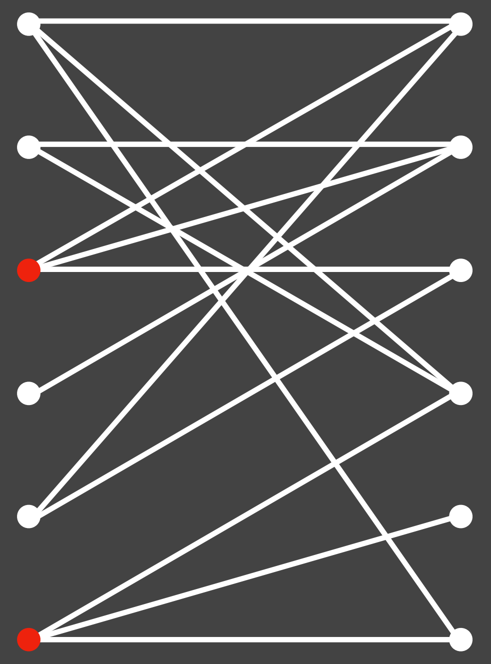
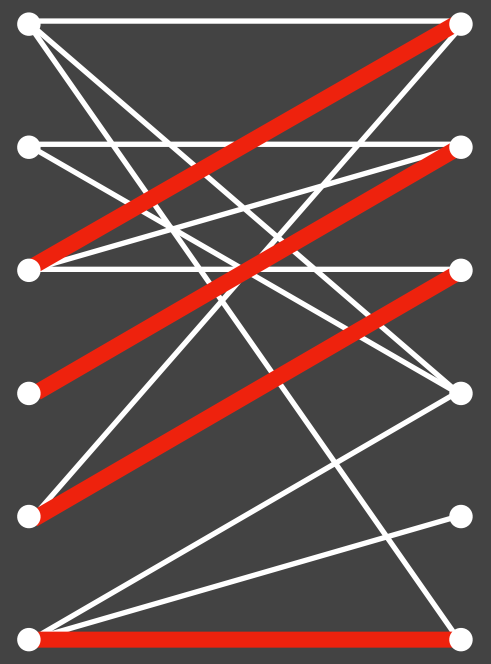


Set Cover

[Gupta Krishnaswamy Kumar Panigrahi 17] [Abboud+ 17] [Bhattacharya Henzinger Nanongkai 19] [Gupta **L.** 20] [Bhattacharya Henzinger Nanongkai Wu 21] [Assadi Solomon21]



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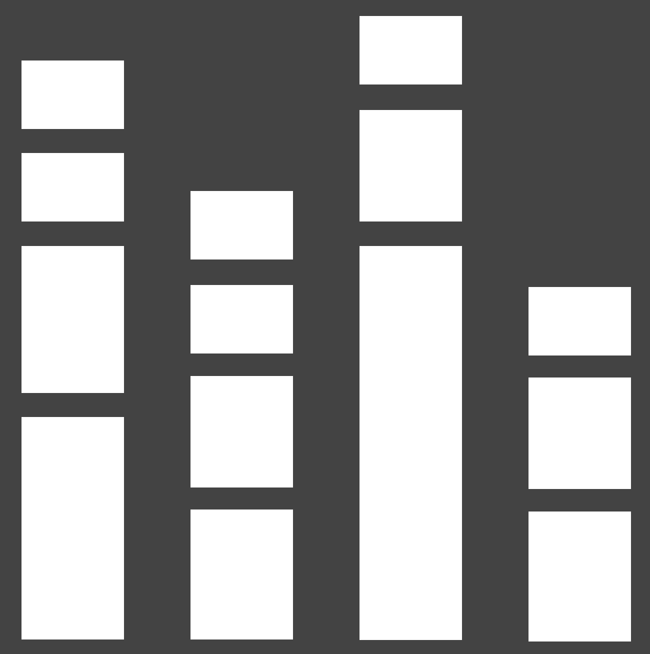
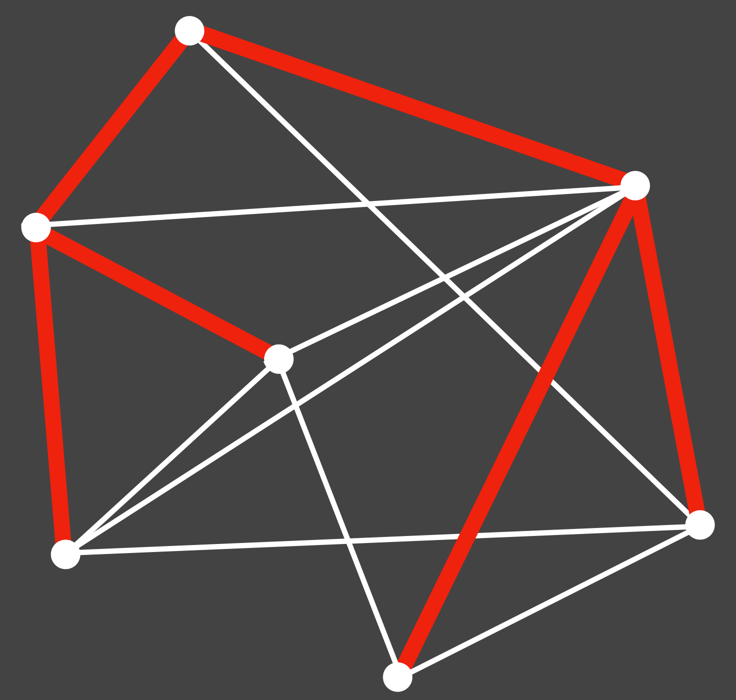
[Folklore] [Grove Kao Krishnan Vitter 95] [Chadhuri Daskalakis Kleinberg Lin 09] [Bosek Leniowski Sankowski Zych] [Bernstein Holm Rotenberg 18]



Load Balancing

[Awerbuch Azar Plotkin Warts 01] [Gupta Kumar Stein 14] [Krishnaswamy Li Suriyanarayana 23]



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General recipe for designing low-recourse algorithms?
LP Relax-and-Round?

The Meta (Fractional) Problem

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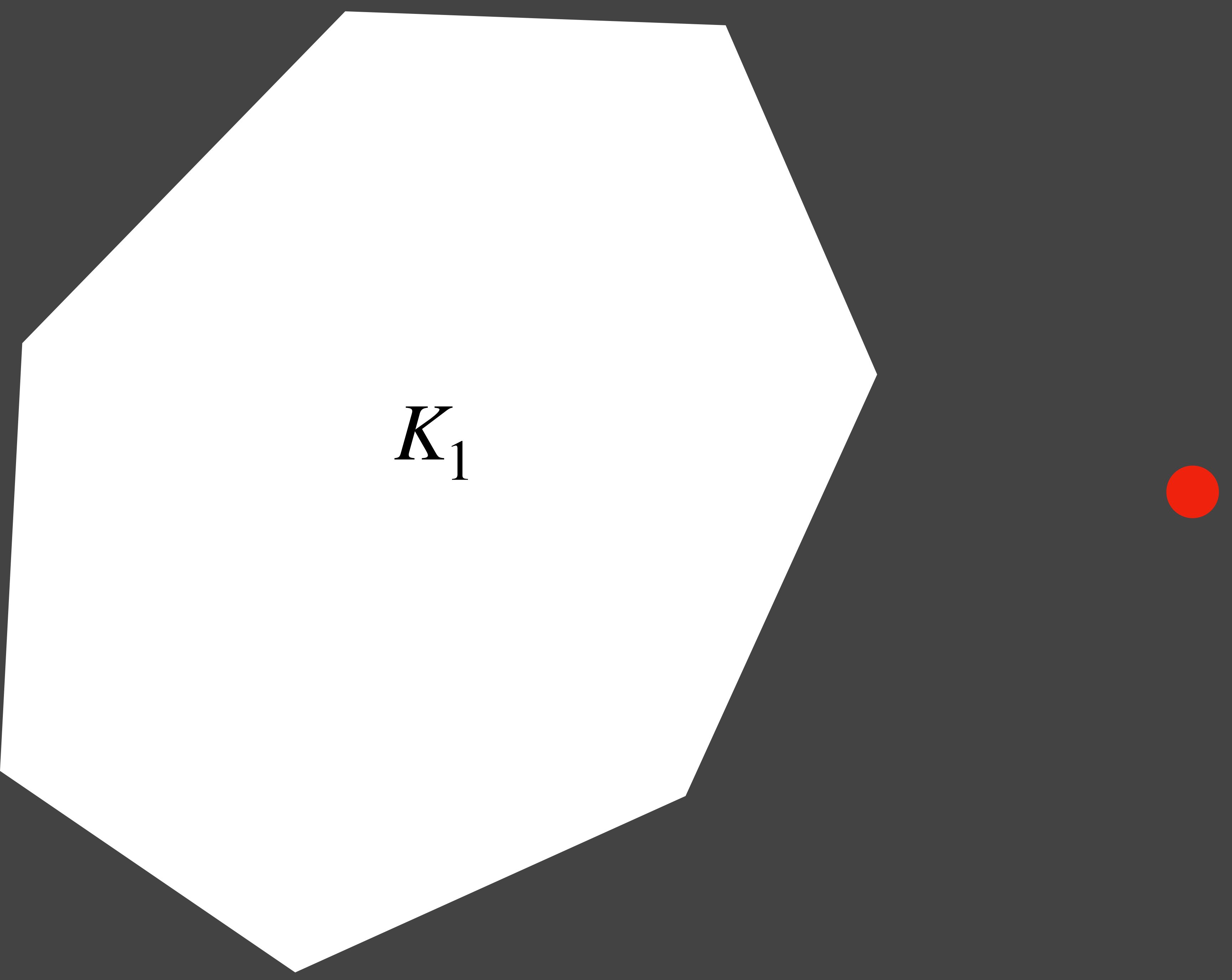

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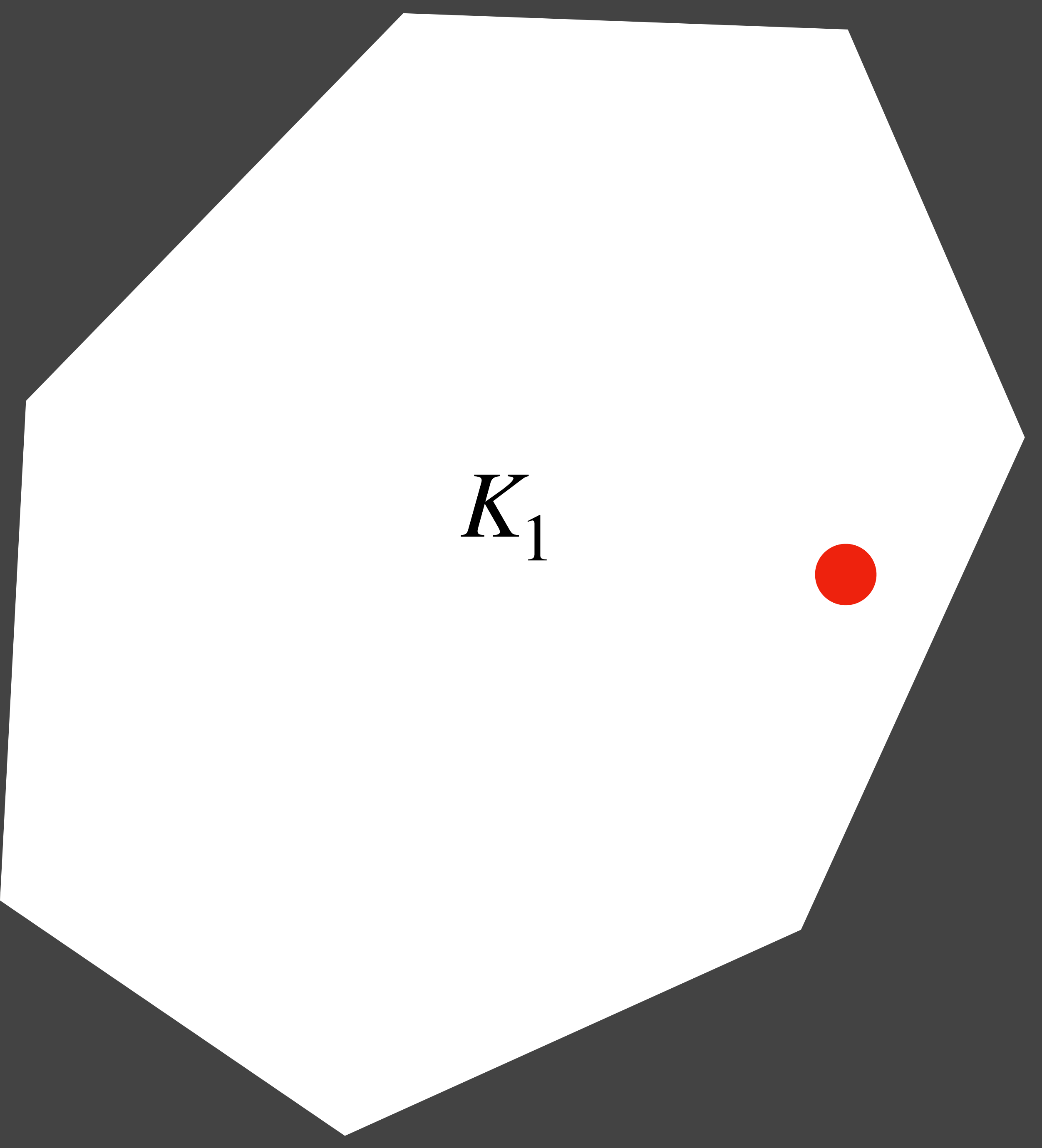

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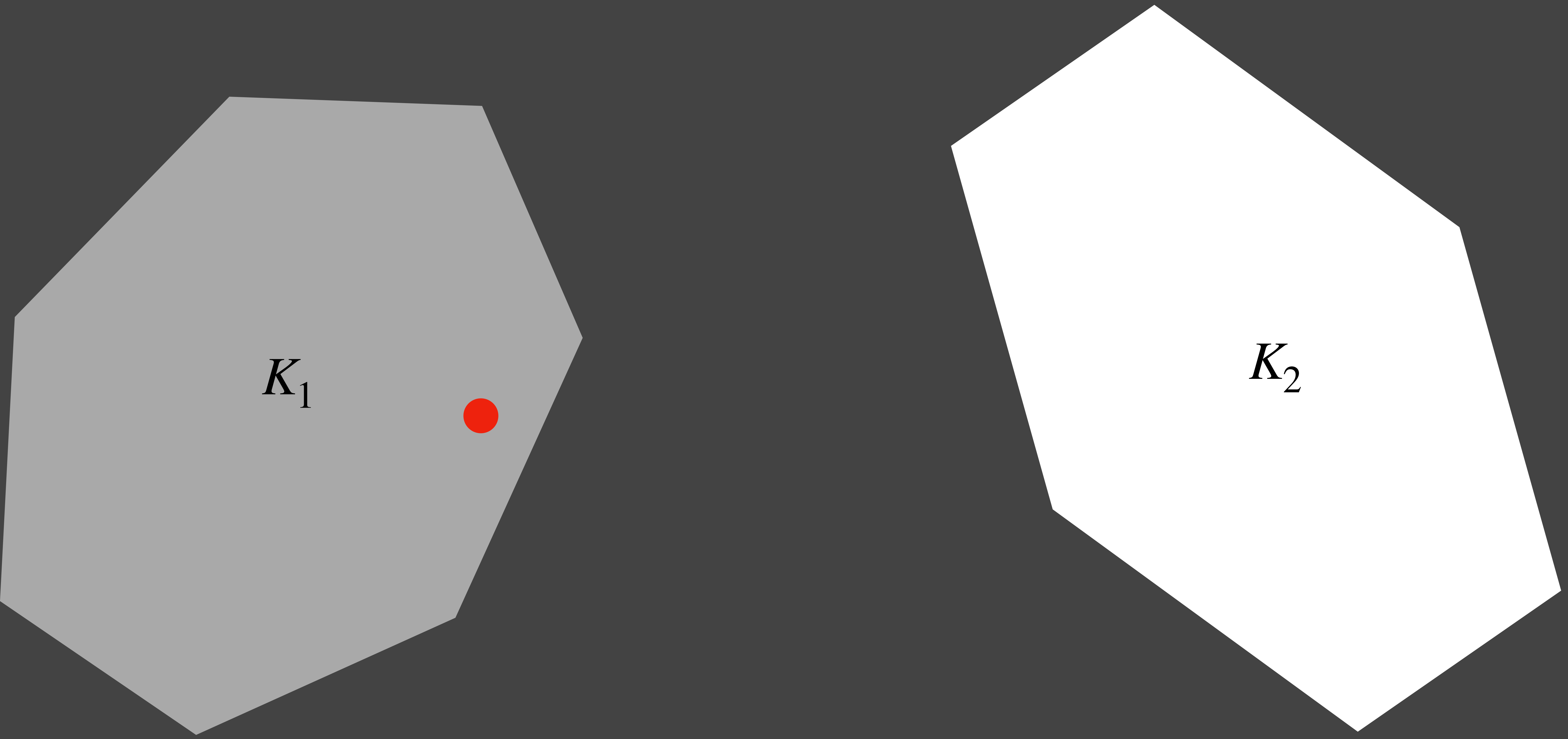

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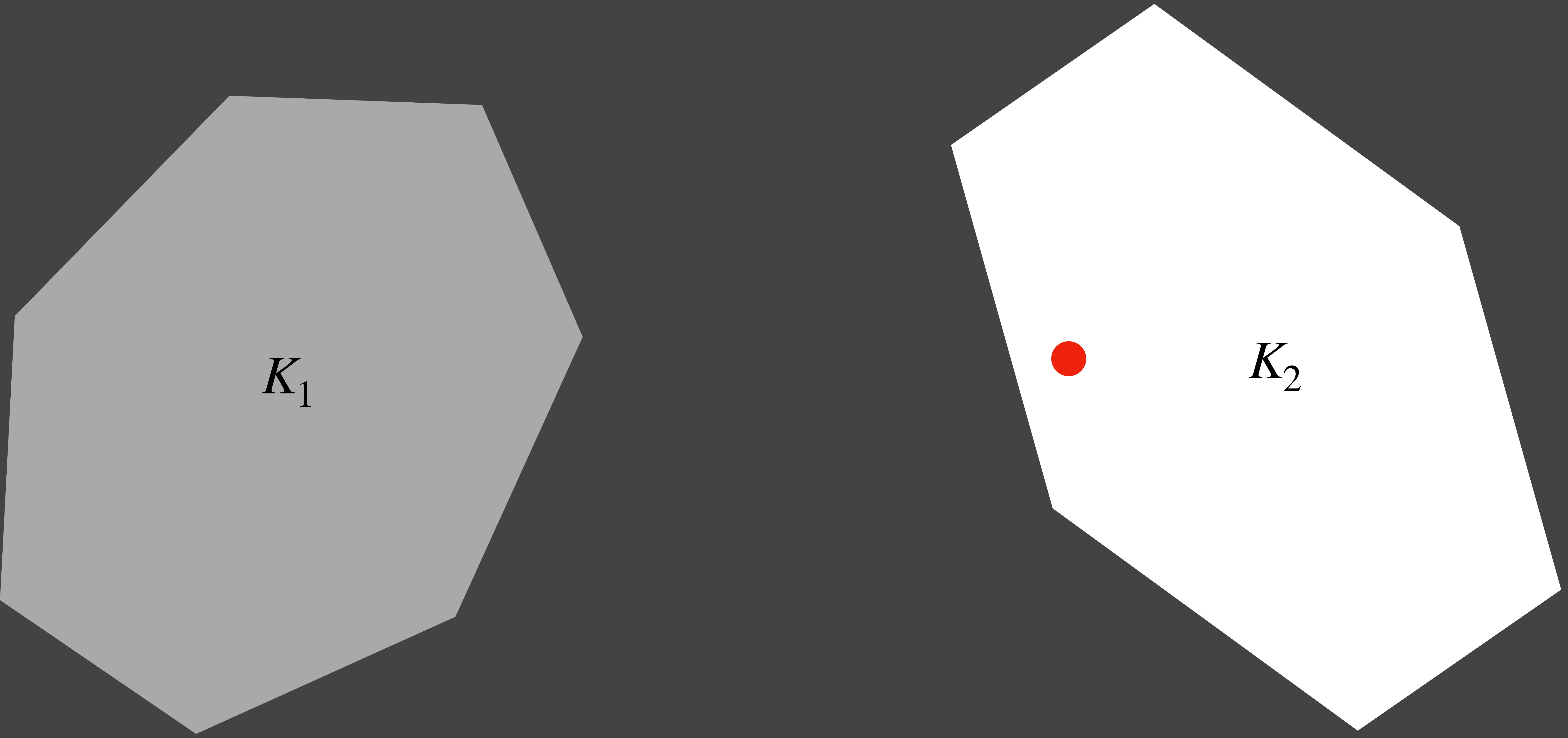

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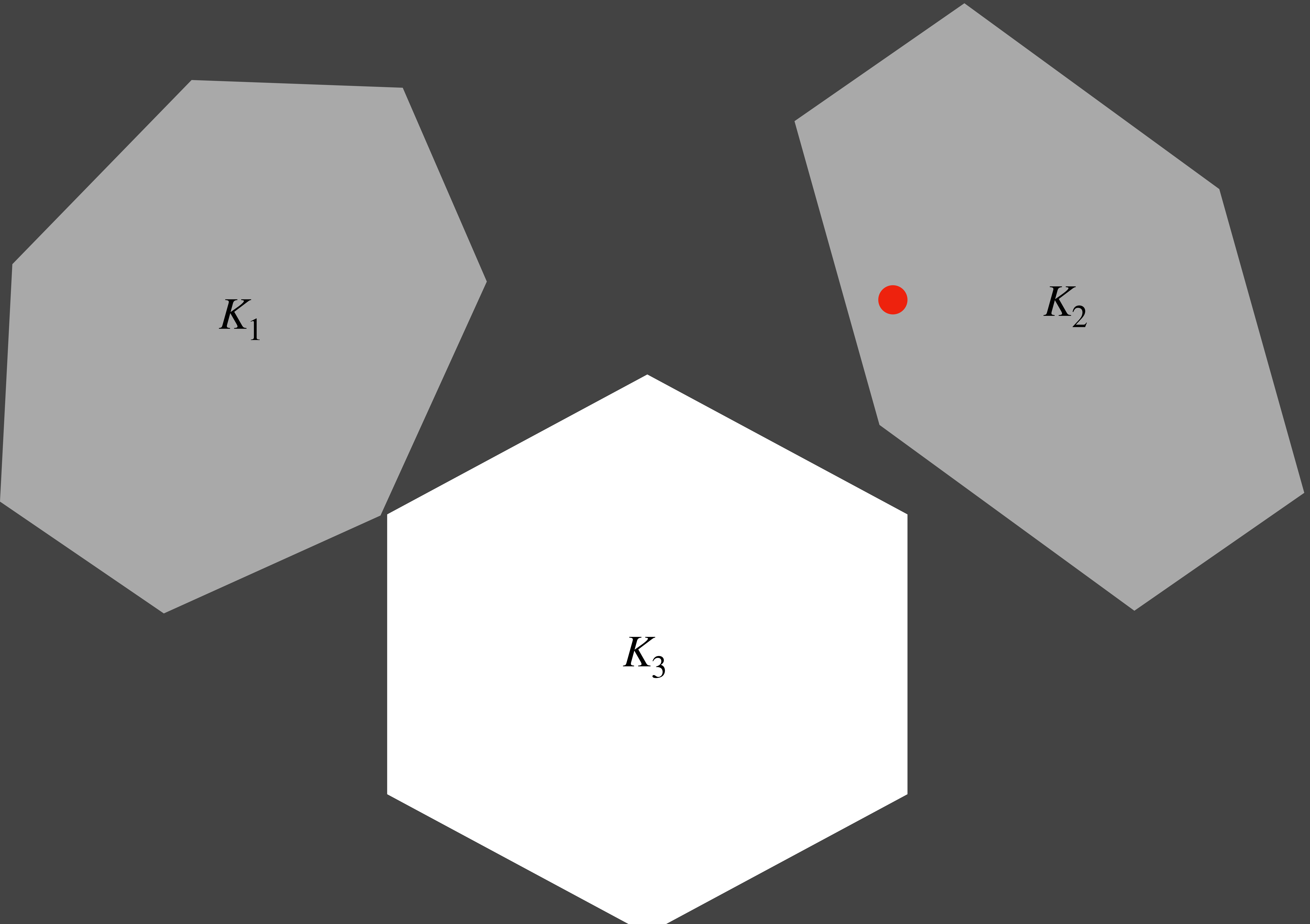

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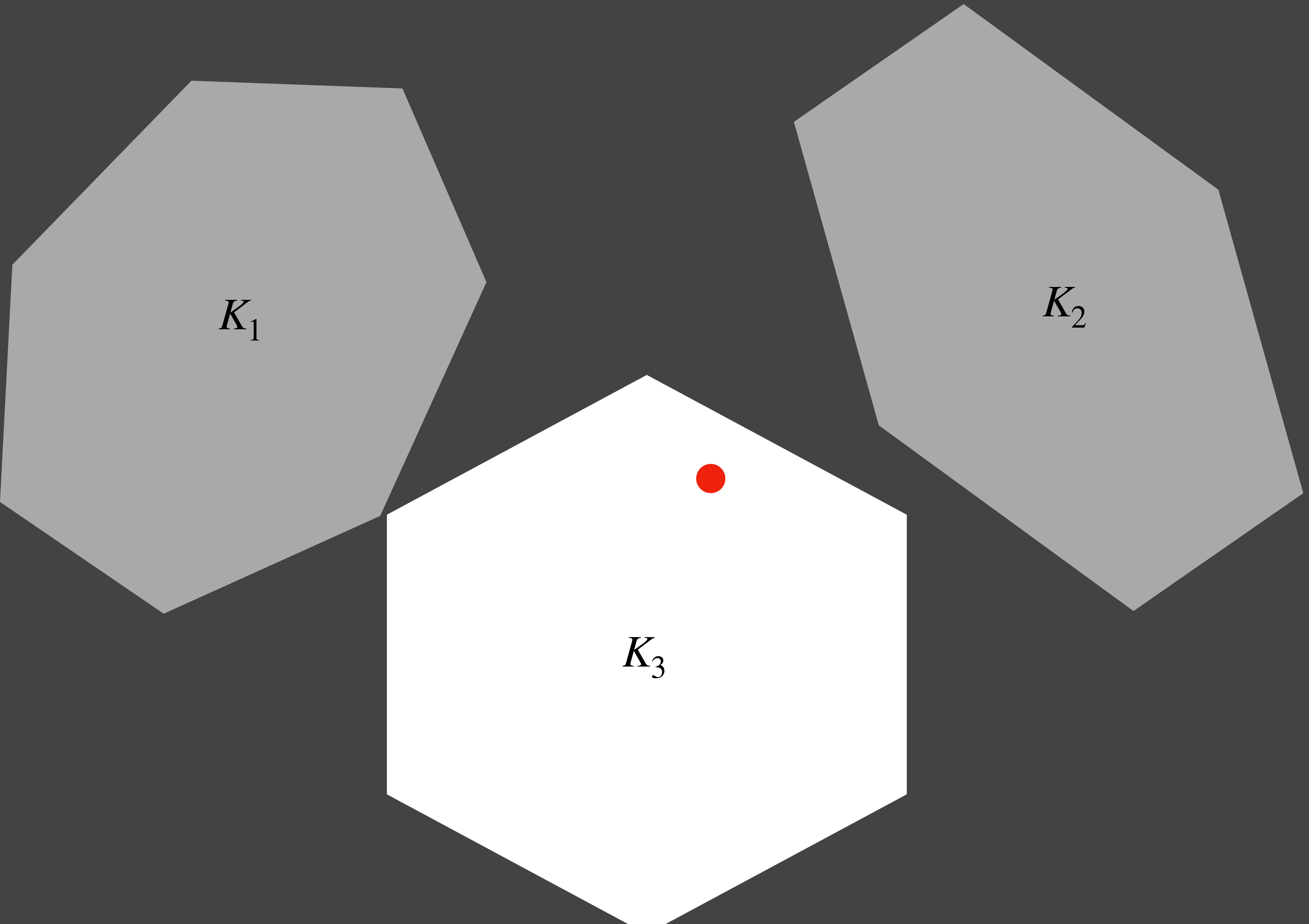

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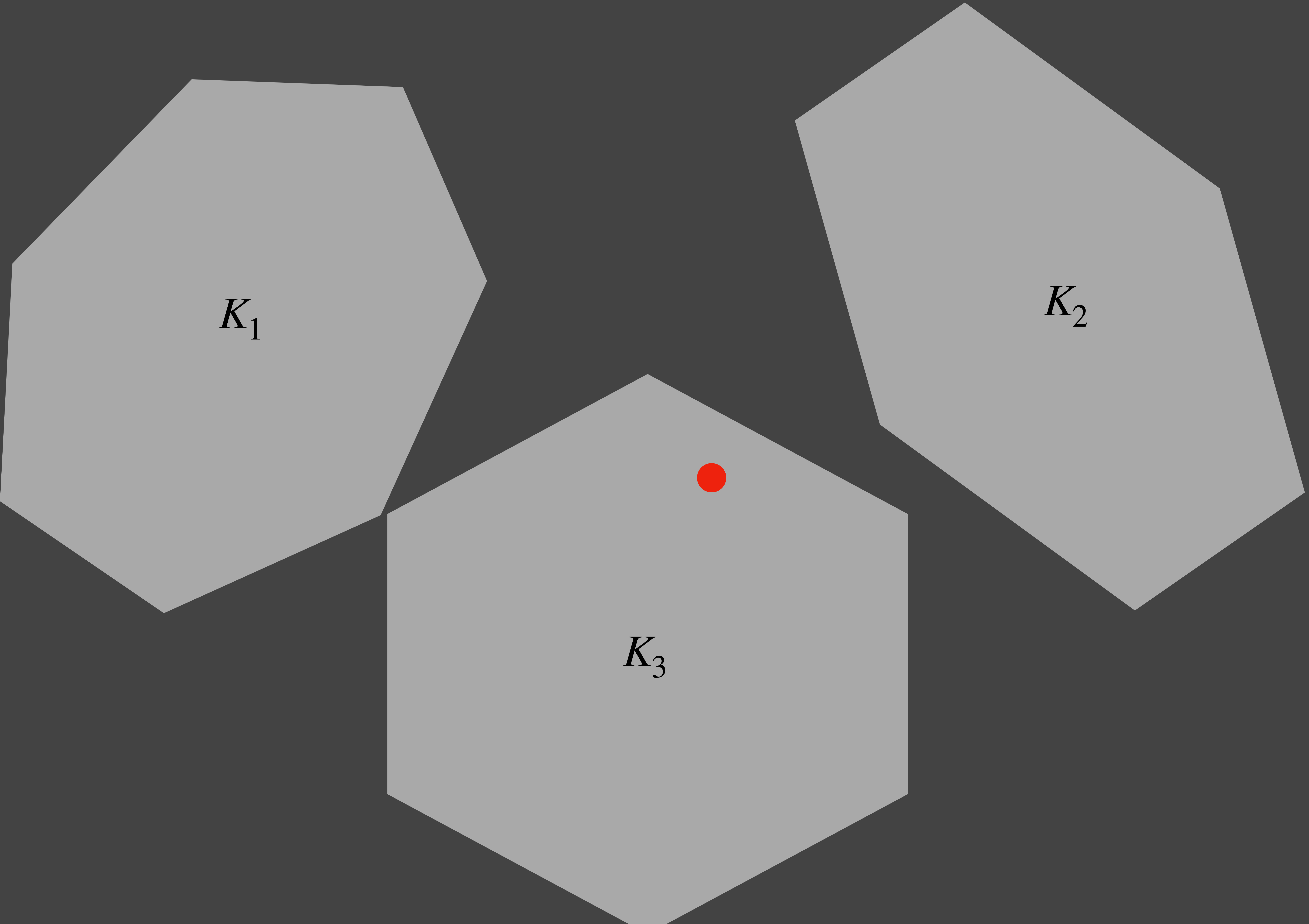

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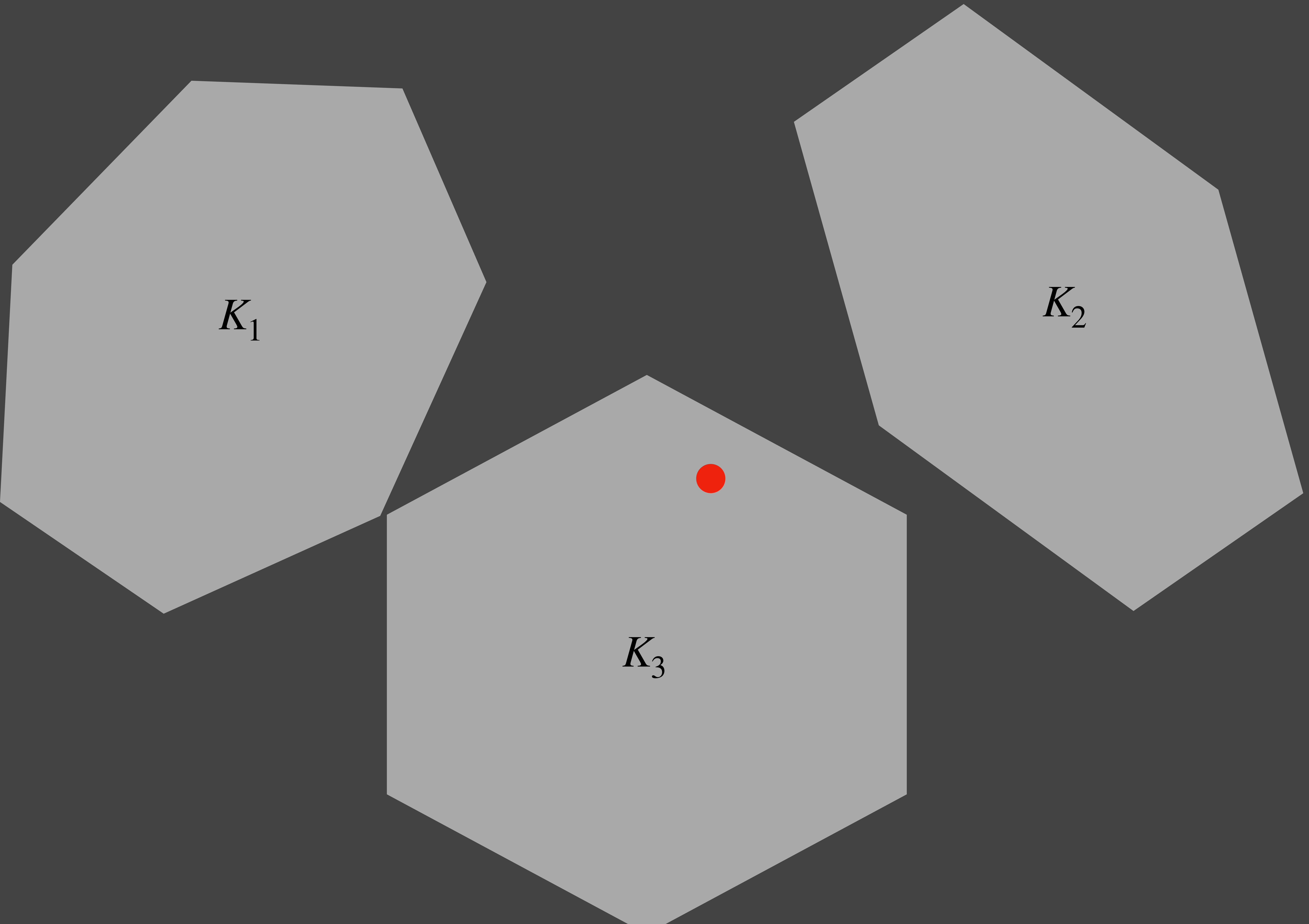

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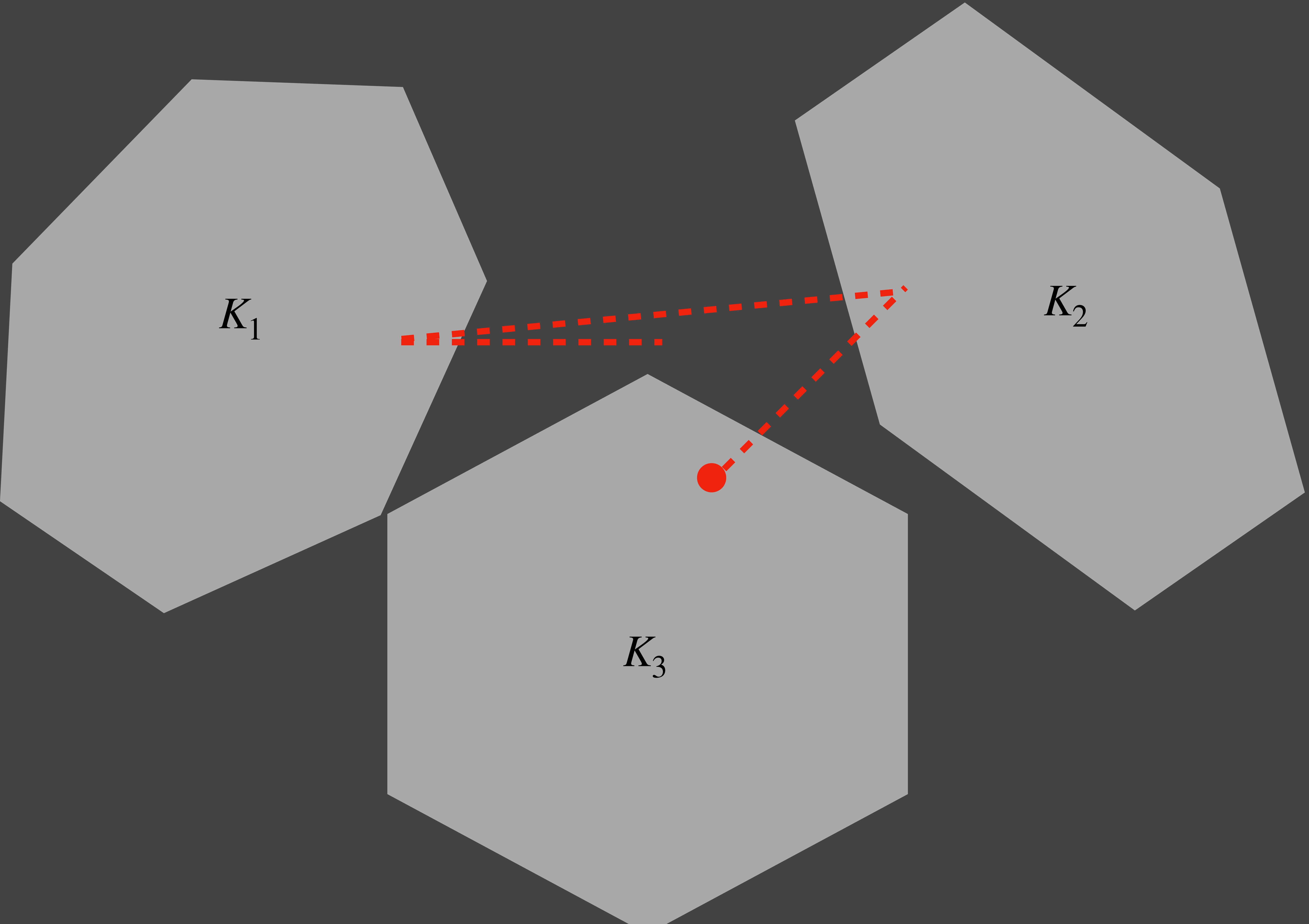

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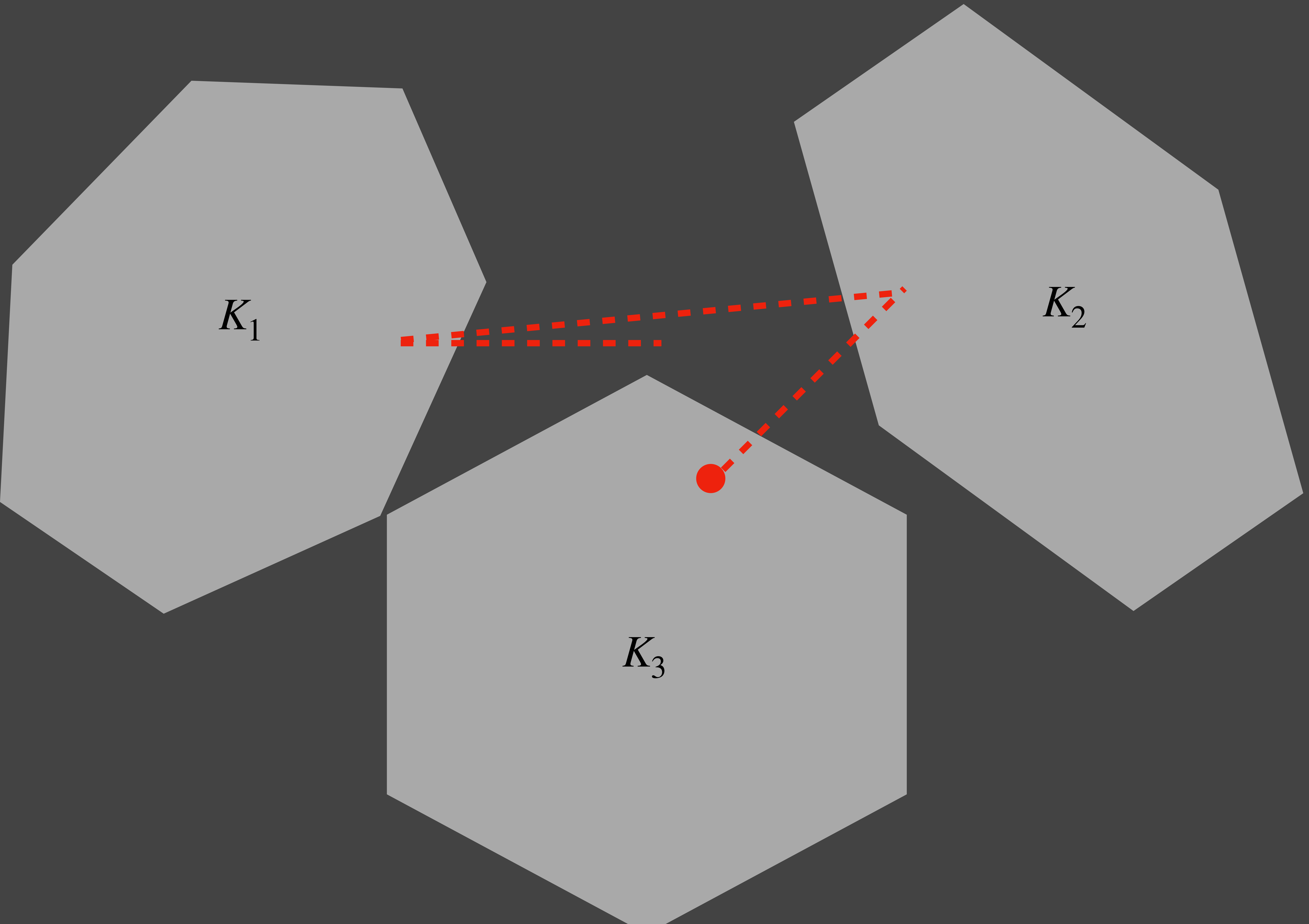

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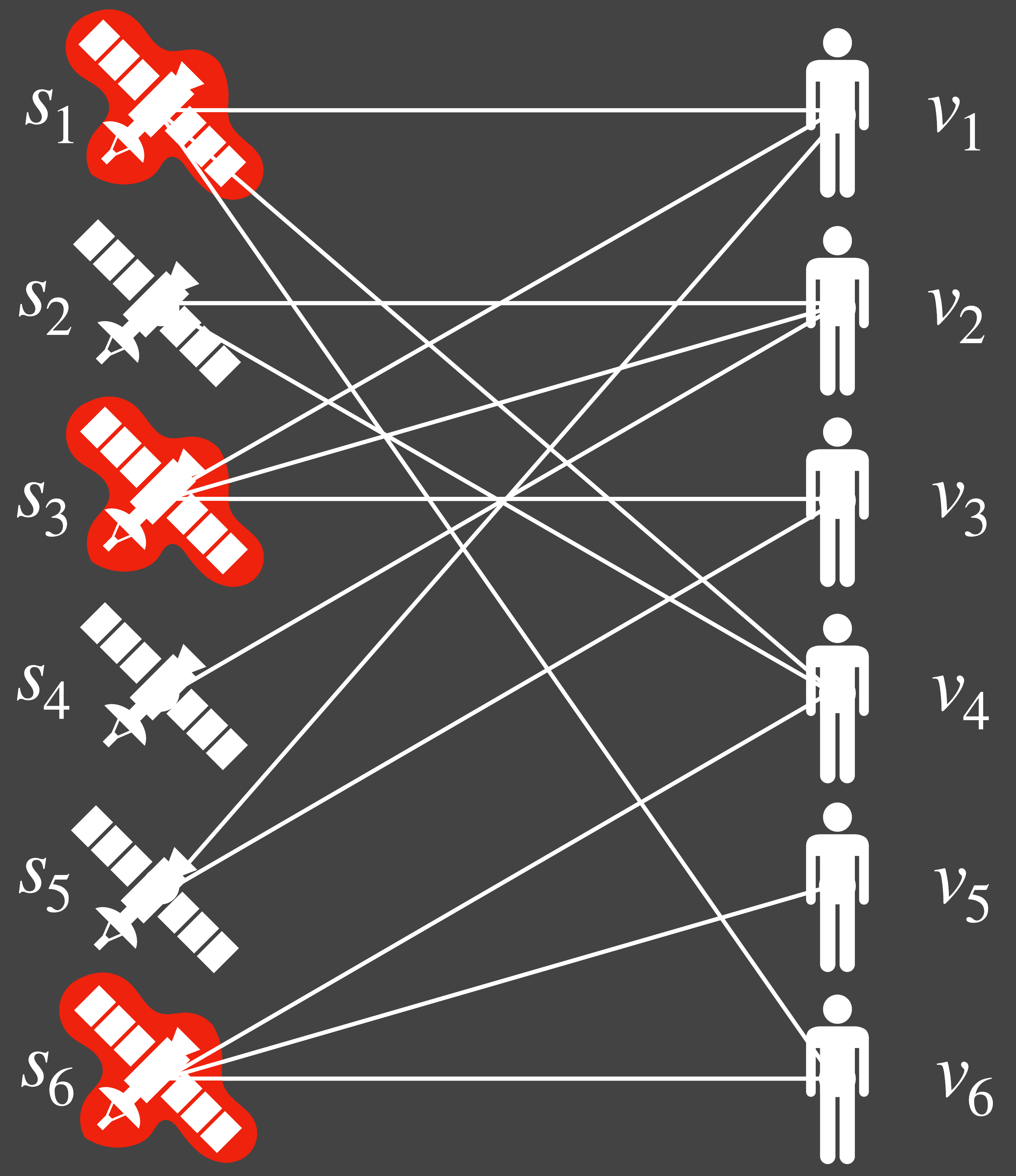

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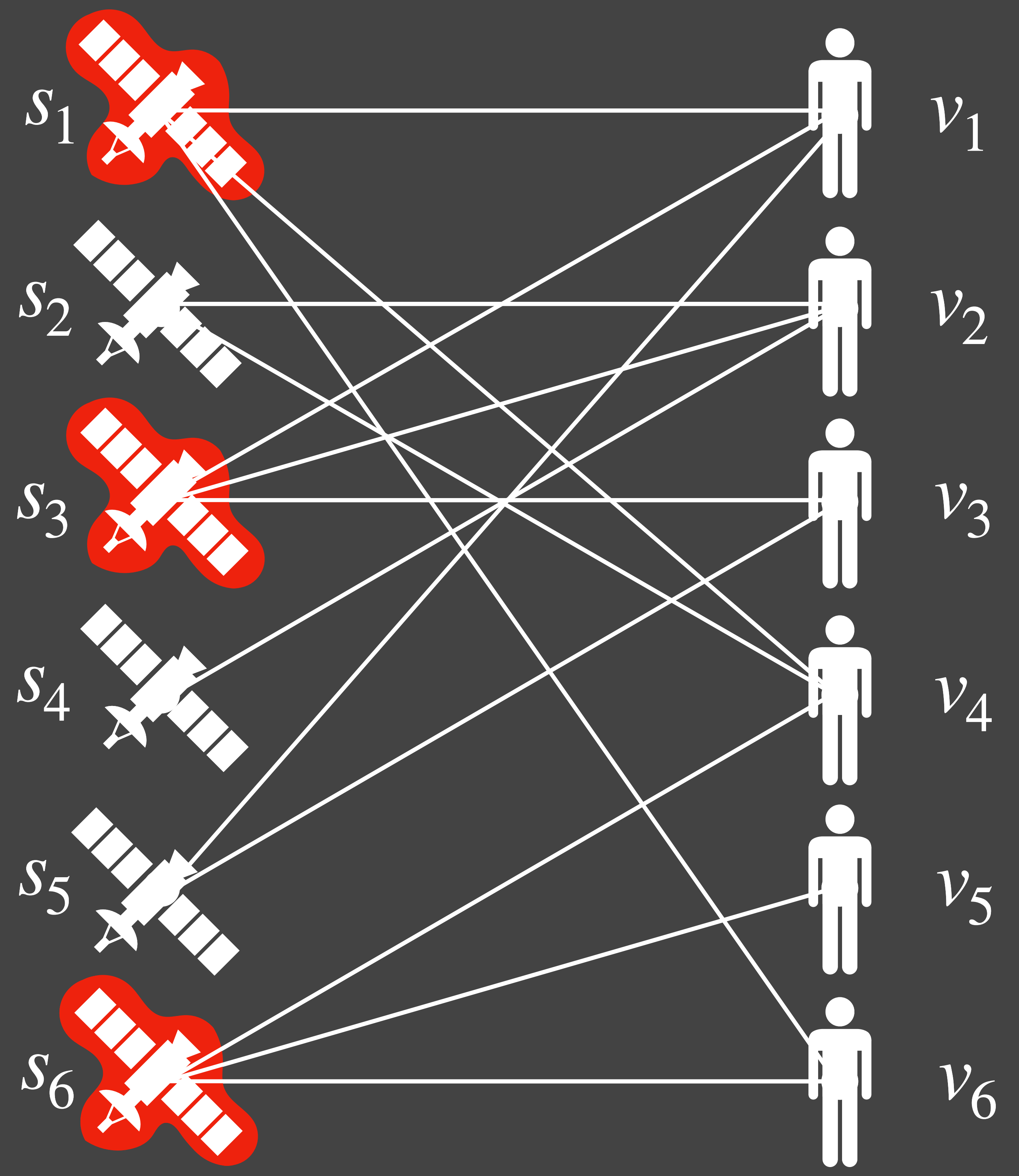

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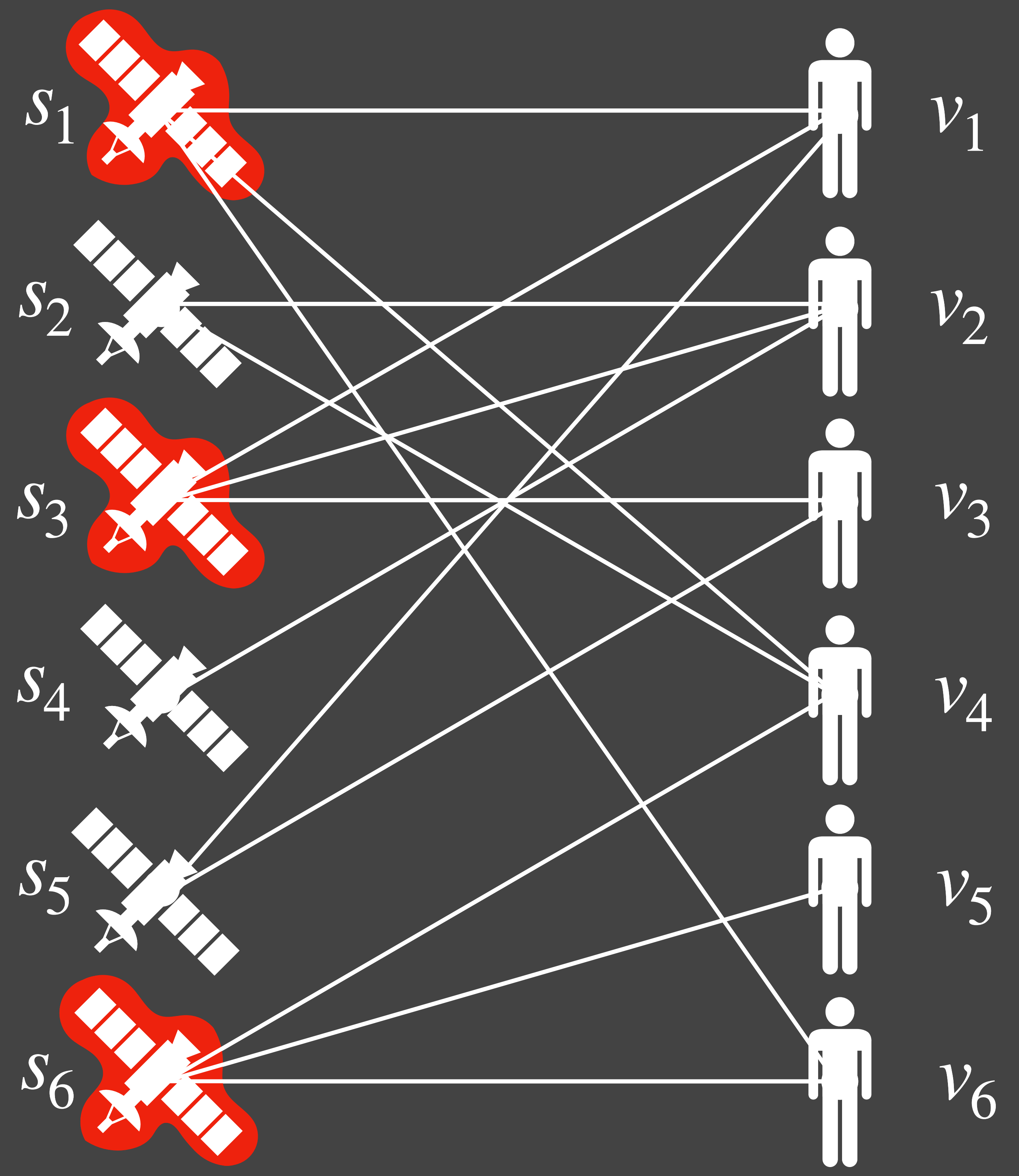

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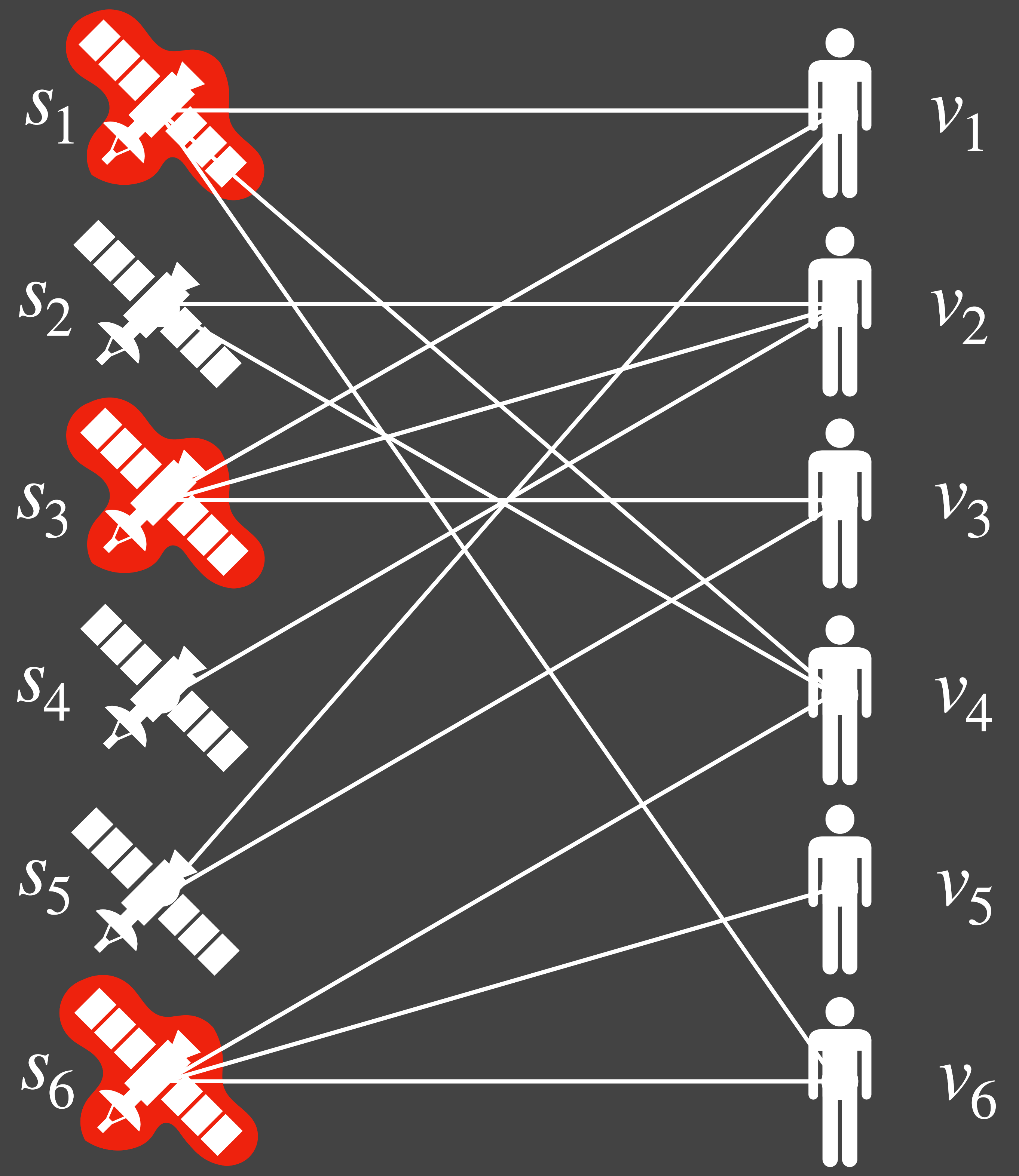

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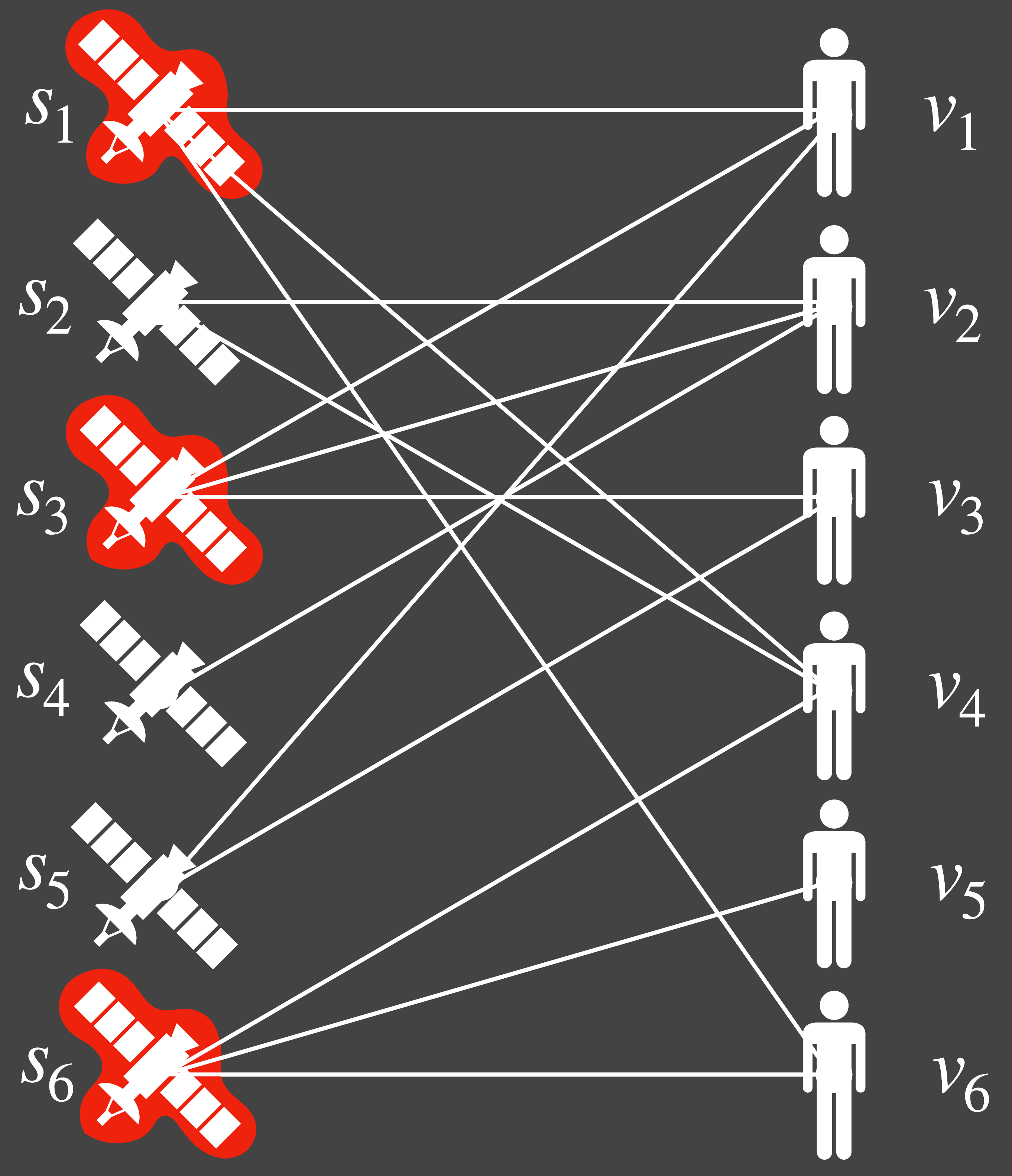

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$$K_t = \left\{ x^t \;\middle|\; \sum_{S \ni e} x_S^t \geq 1 \quad \forall e \in U^t, \quad \sum_S x_S^t \leq \beta \cdot \mathrm{OPT}^t \right\}$$

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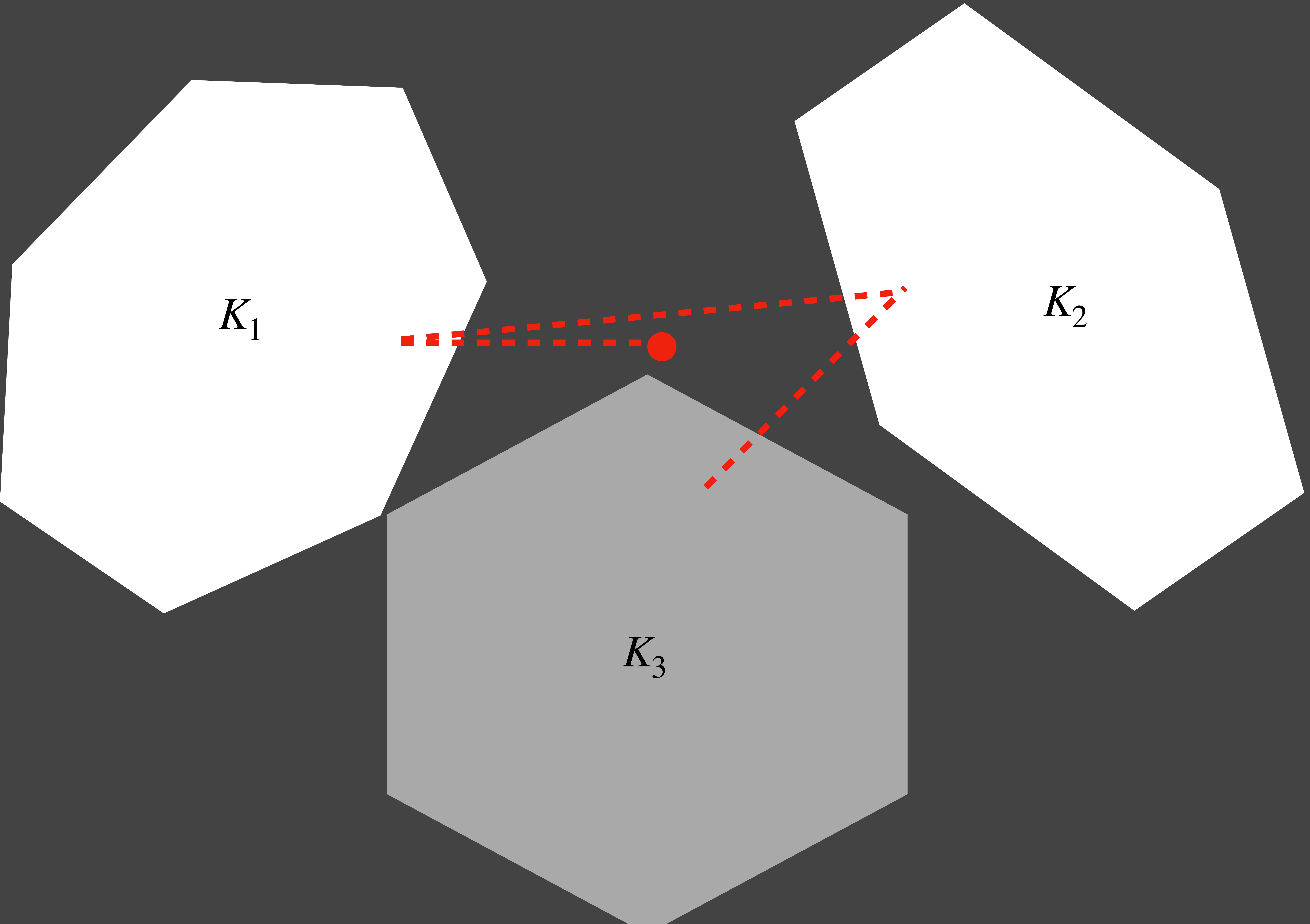

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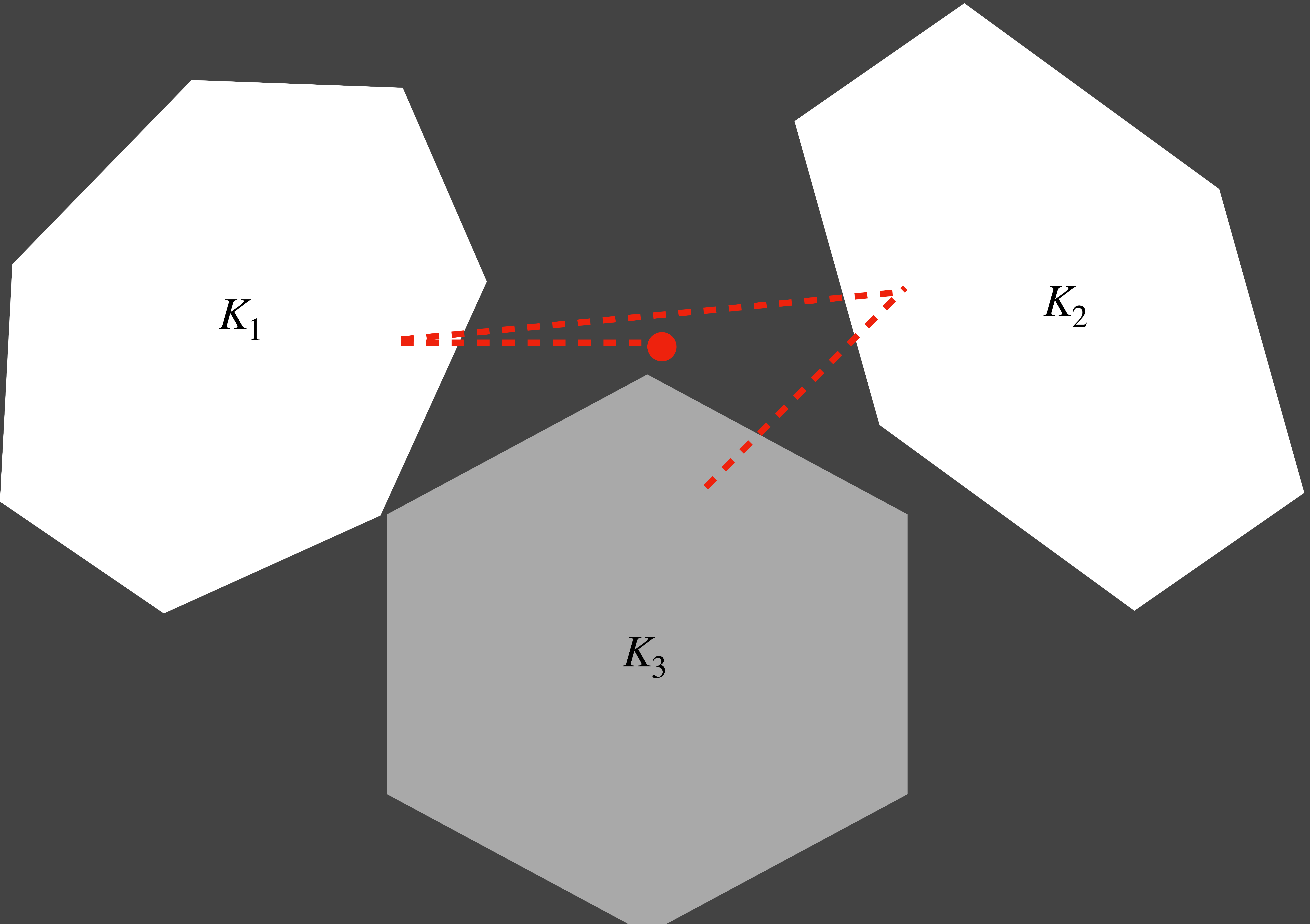

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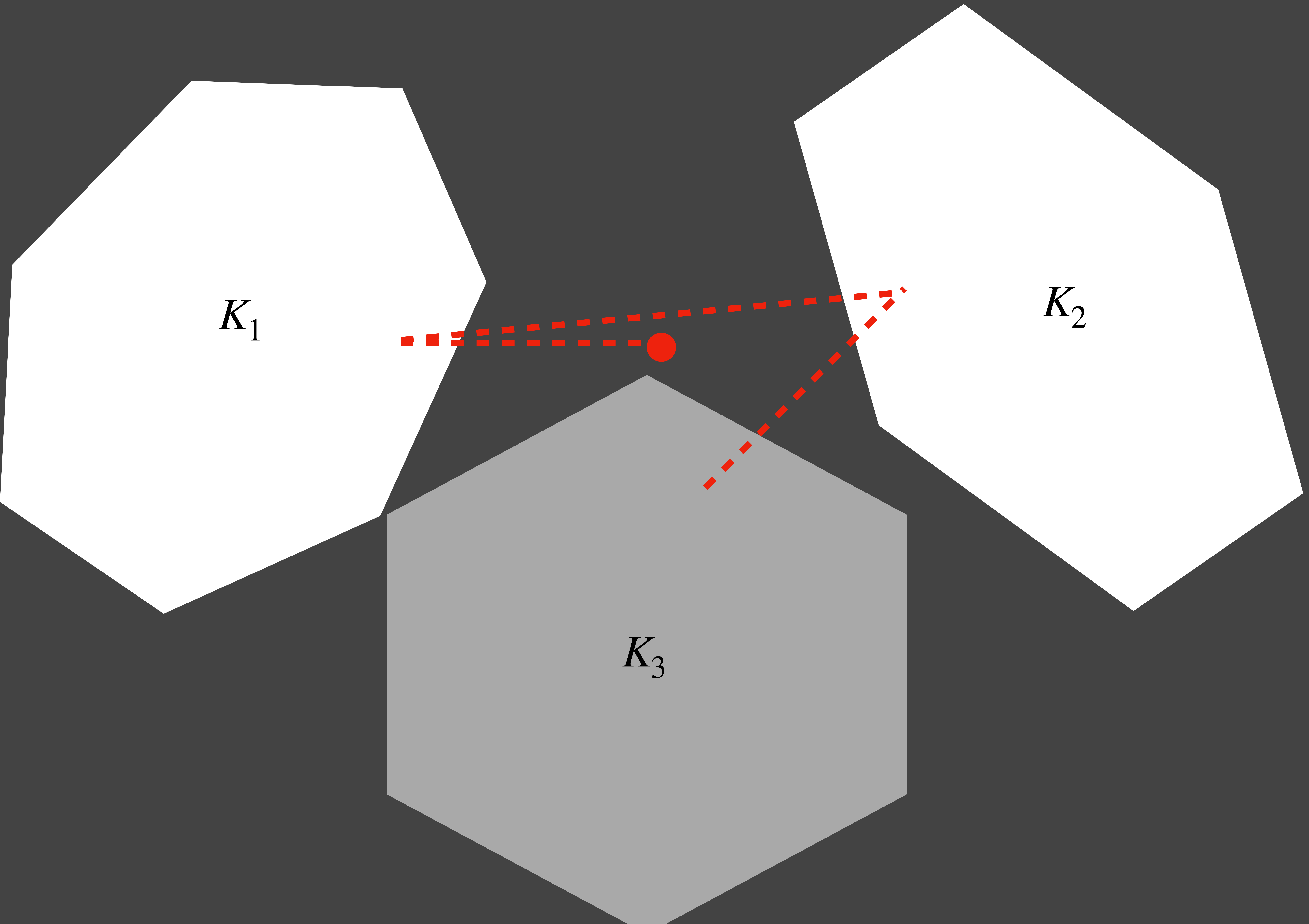

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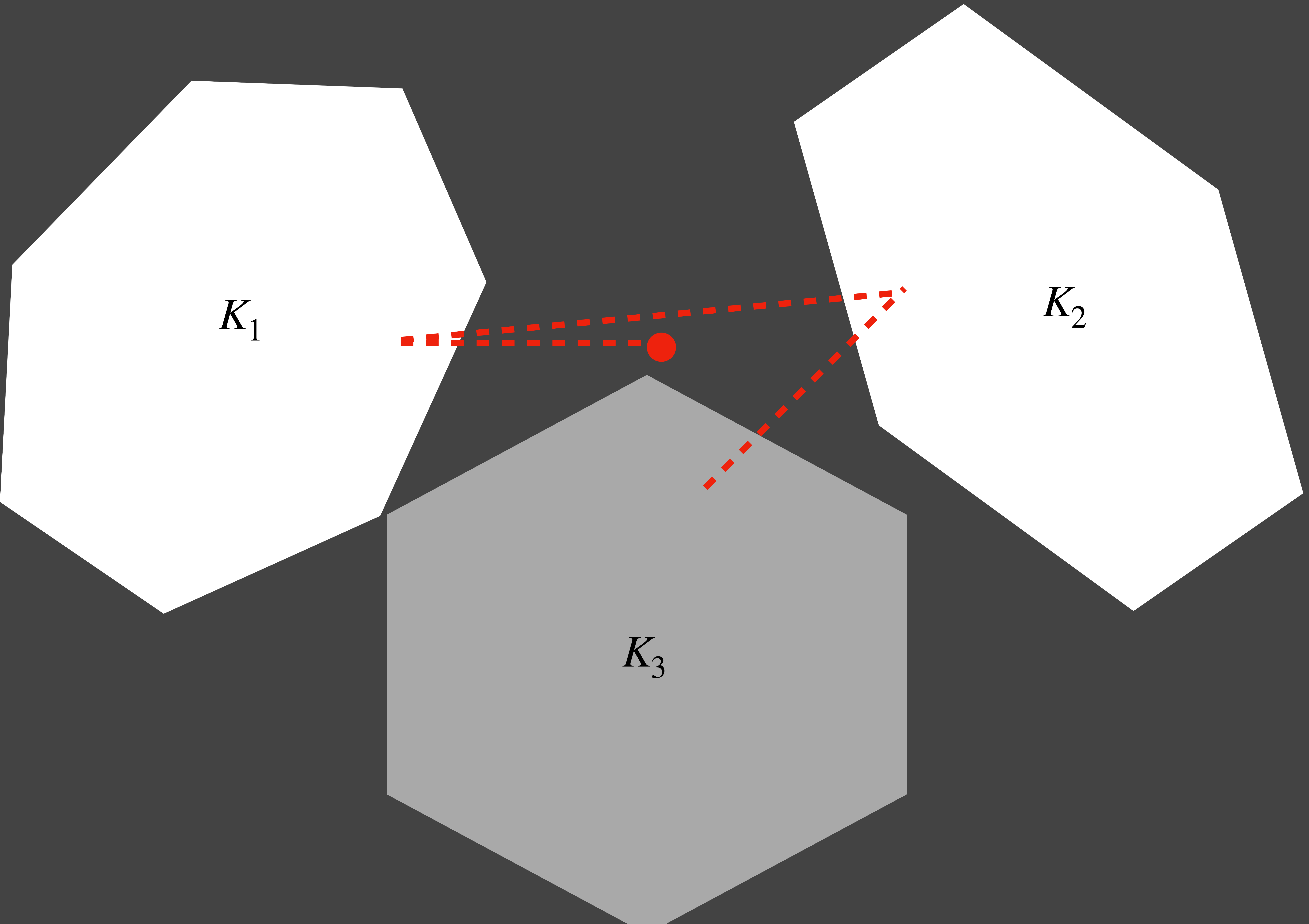

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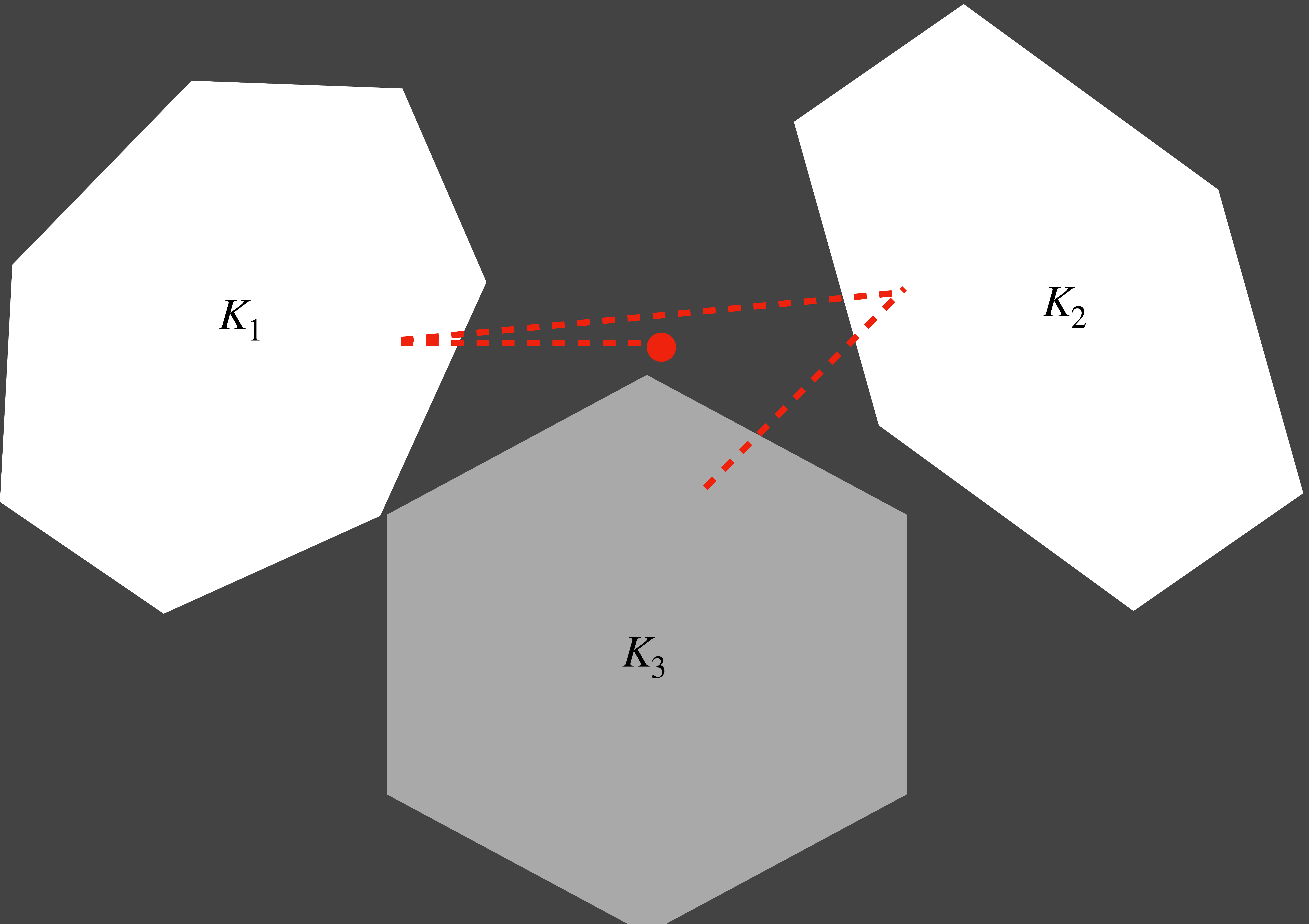

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Too weak for most applications…😰

Can We Exploit More Structure?

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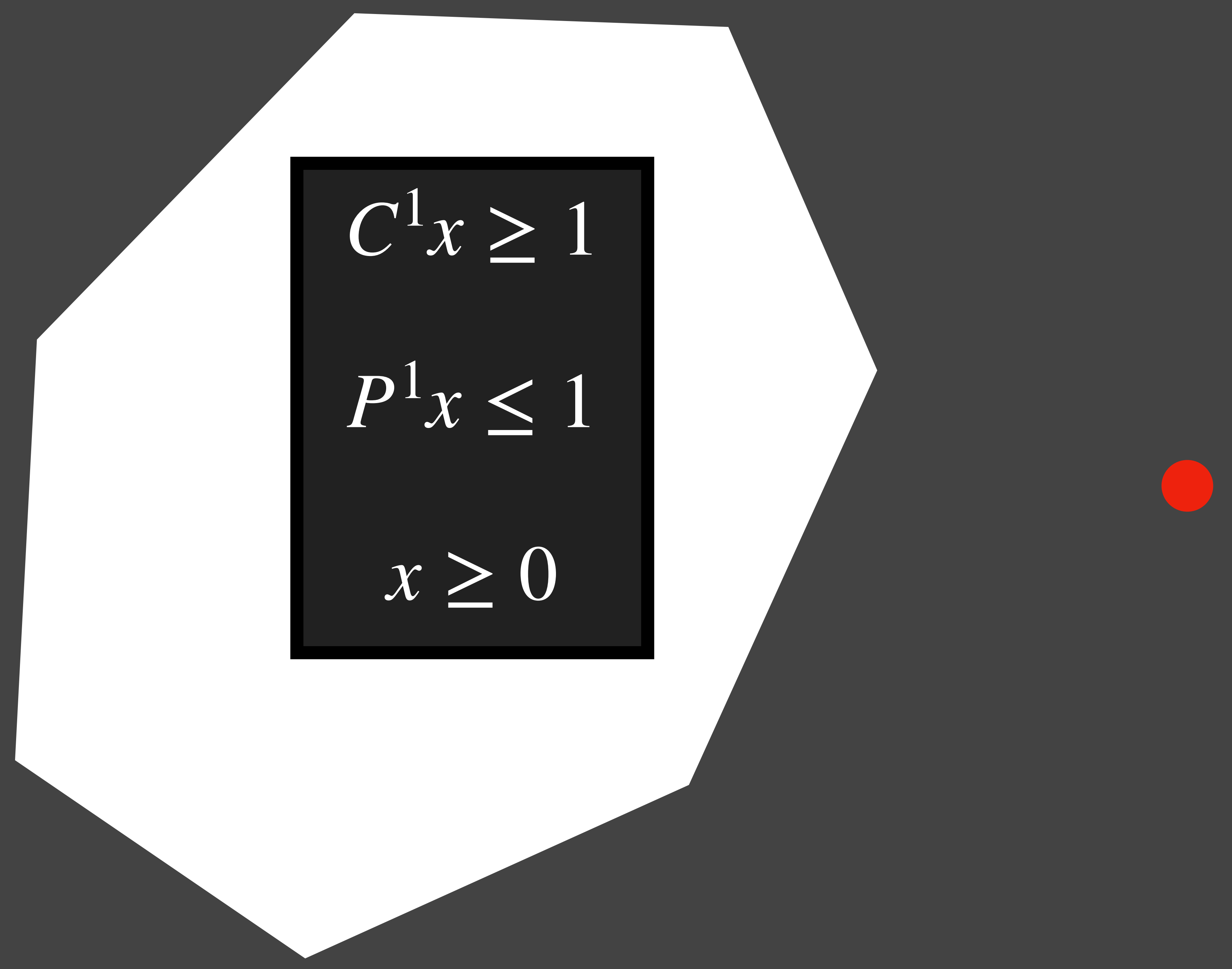

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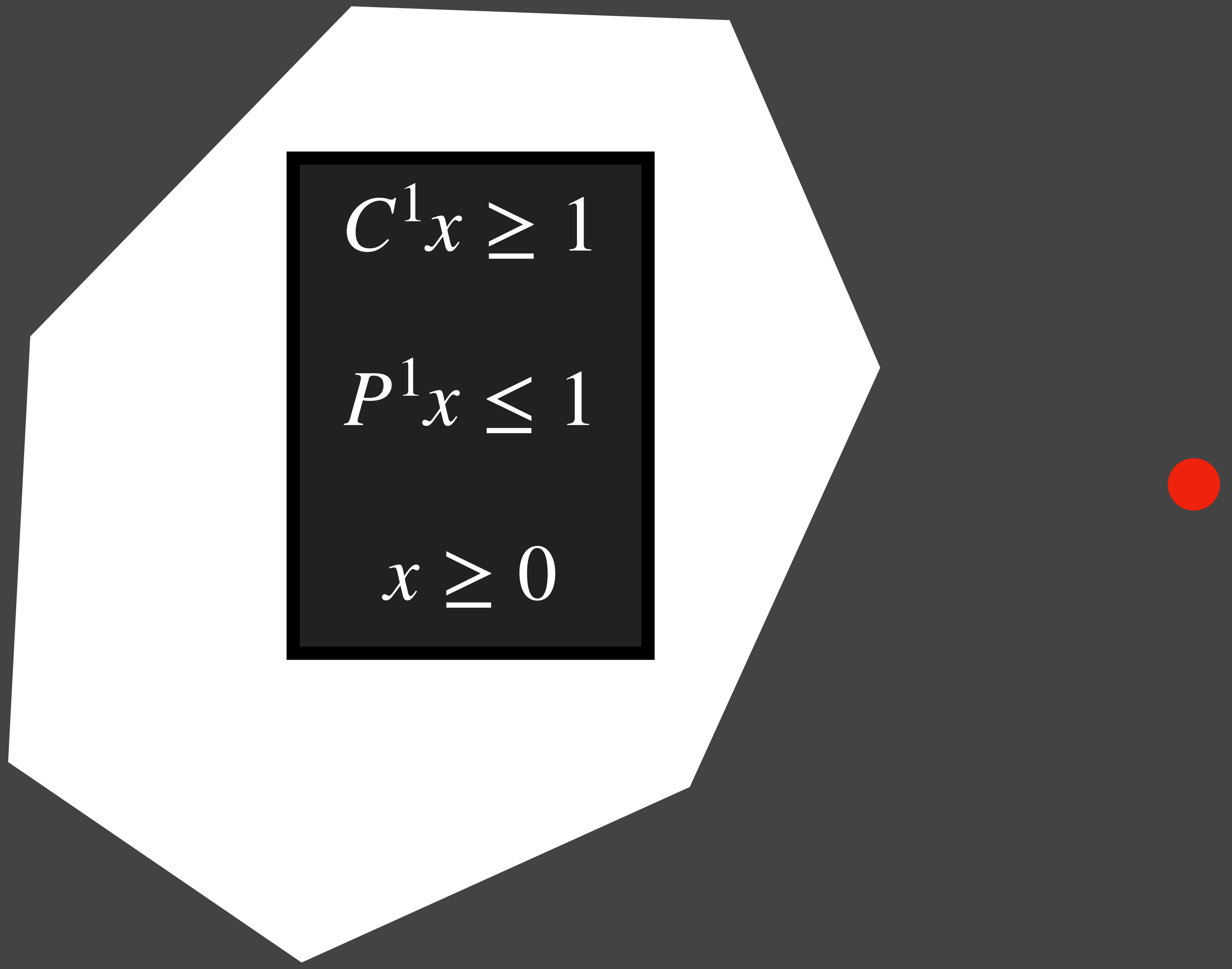

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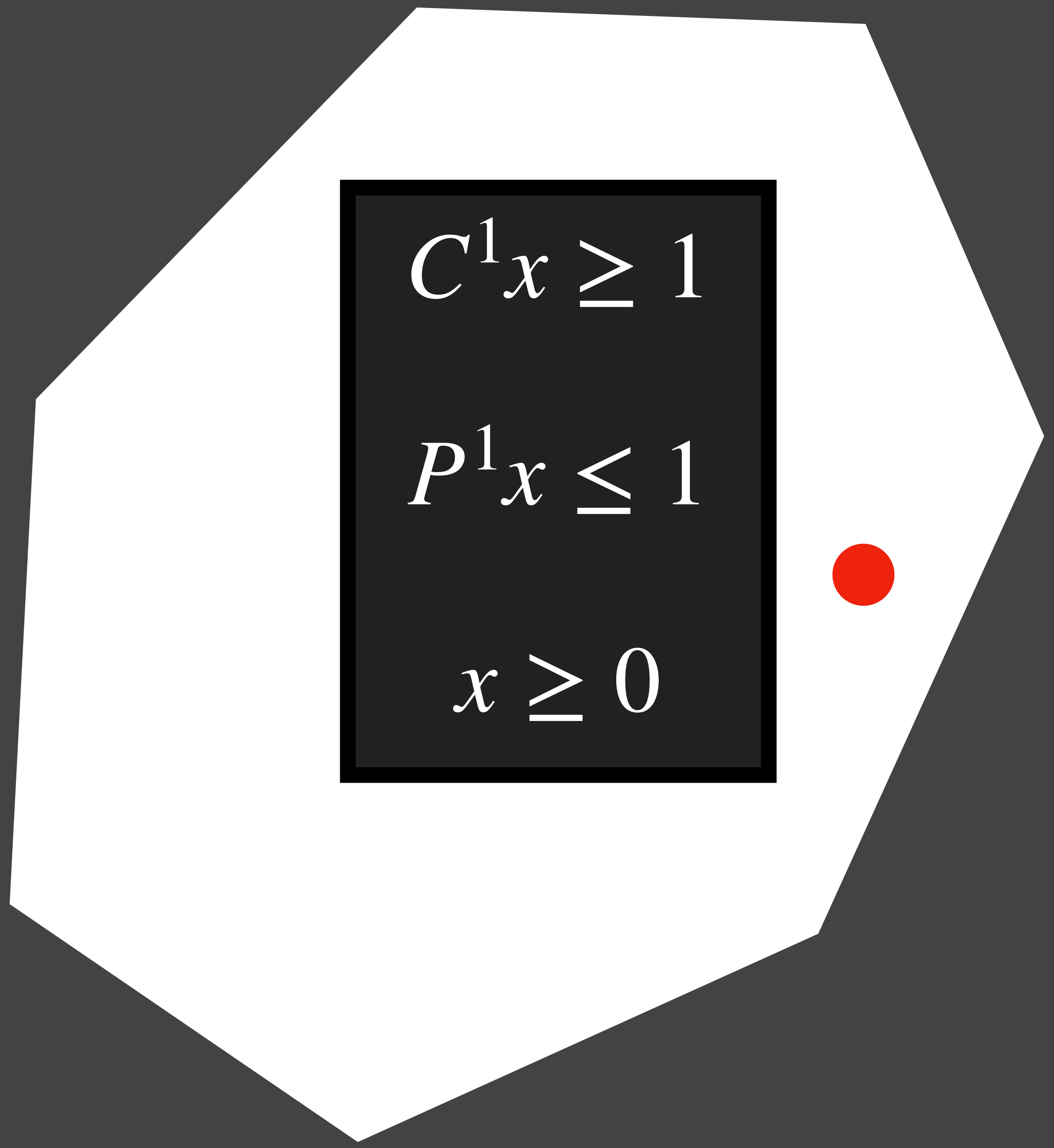

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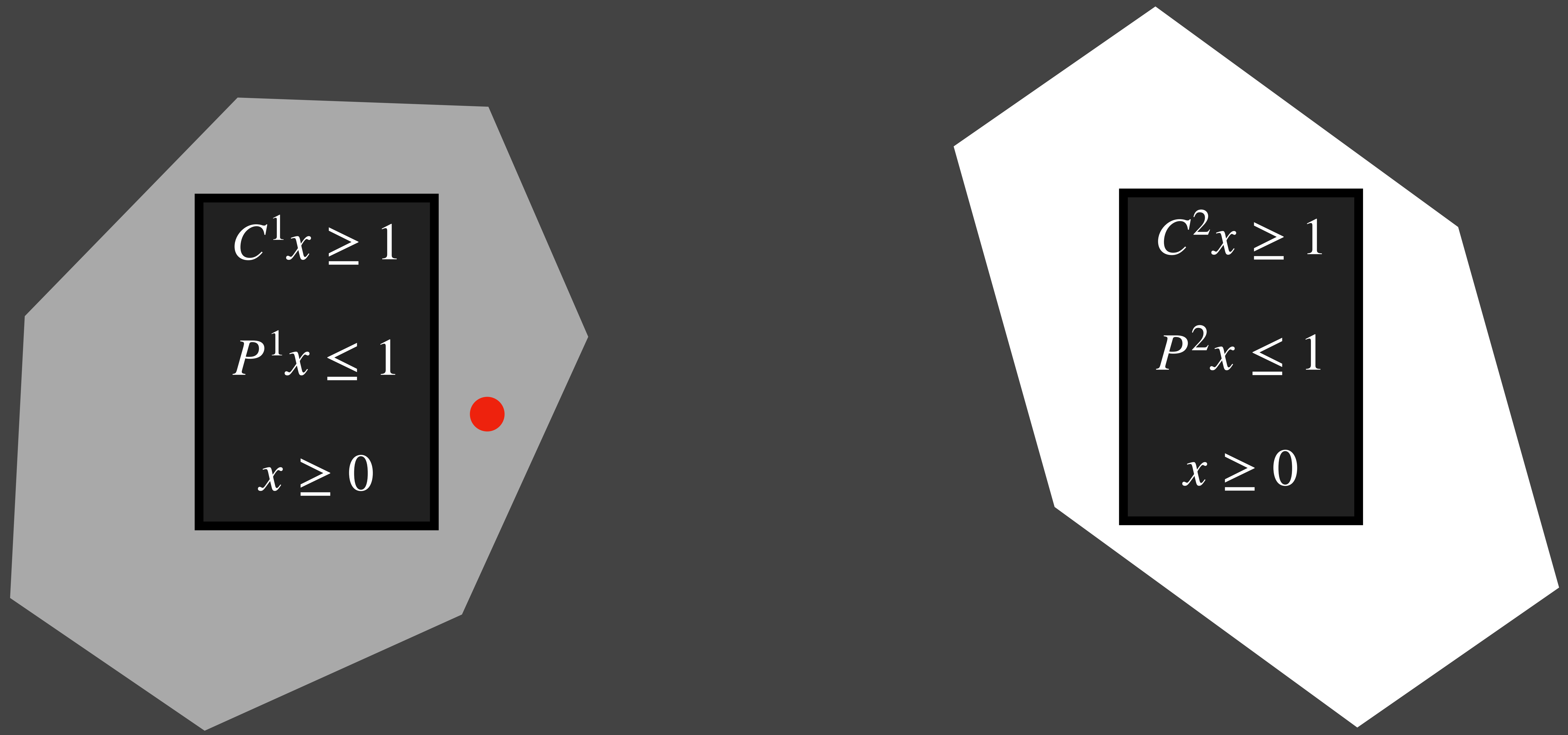

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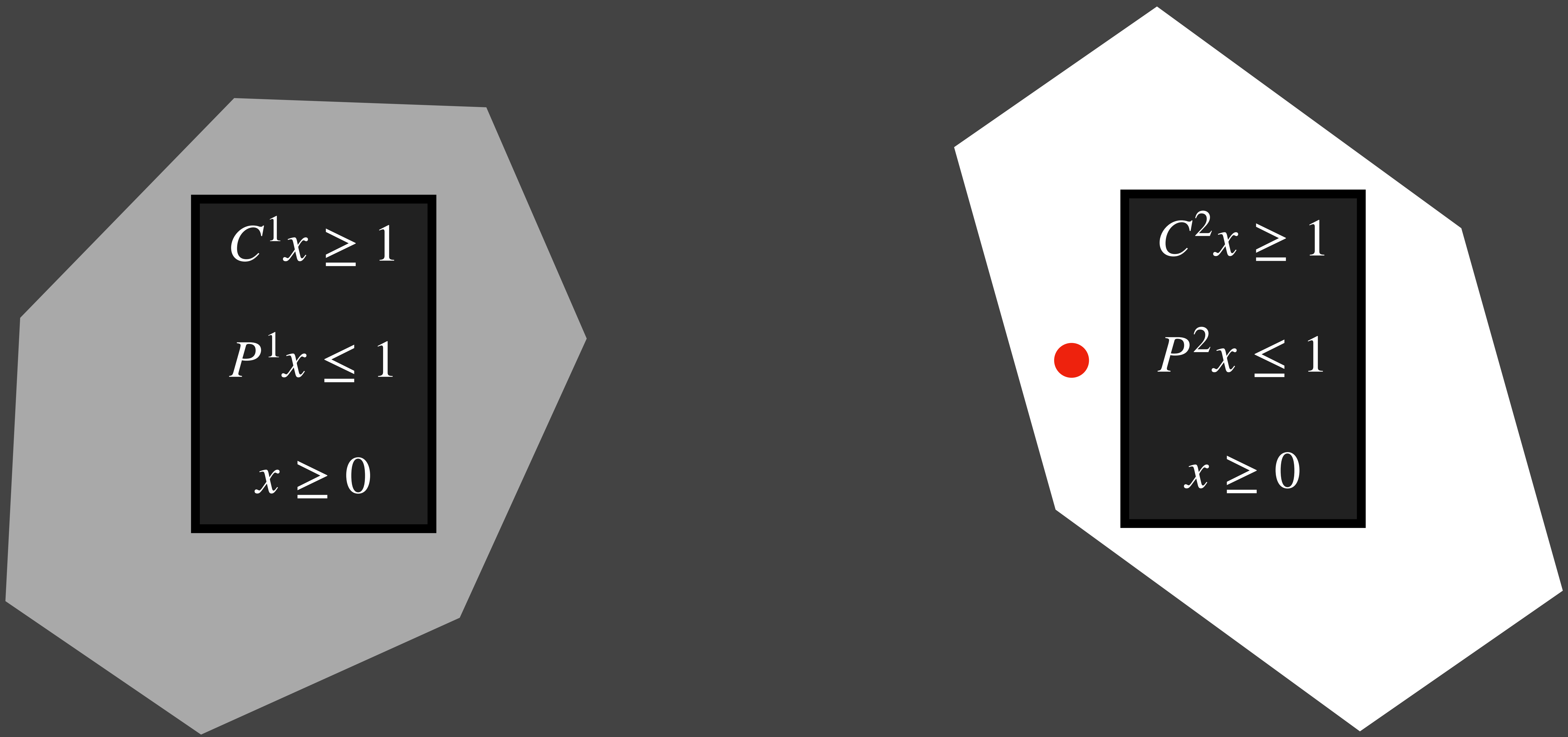

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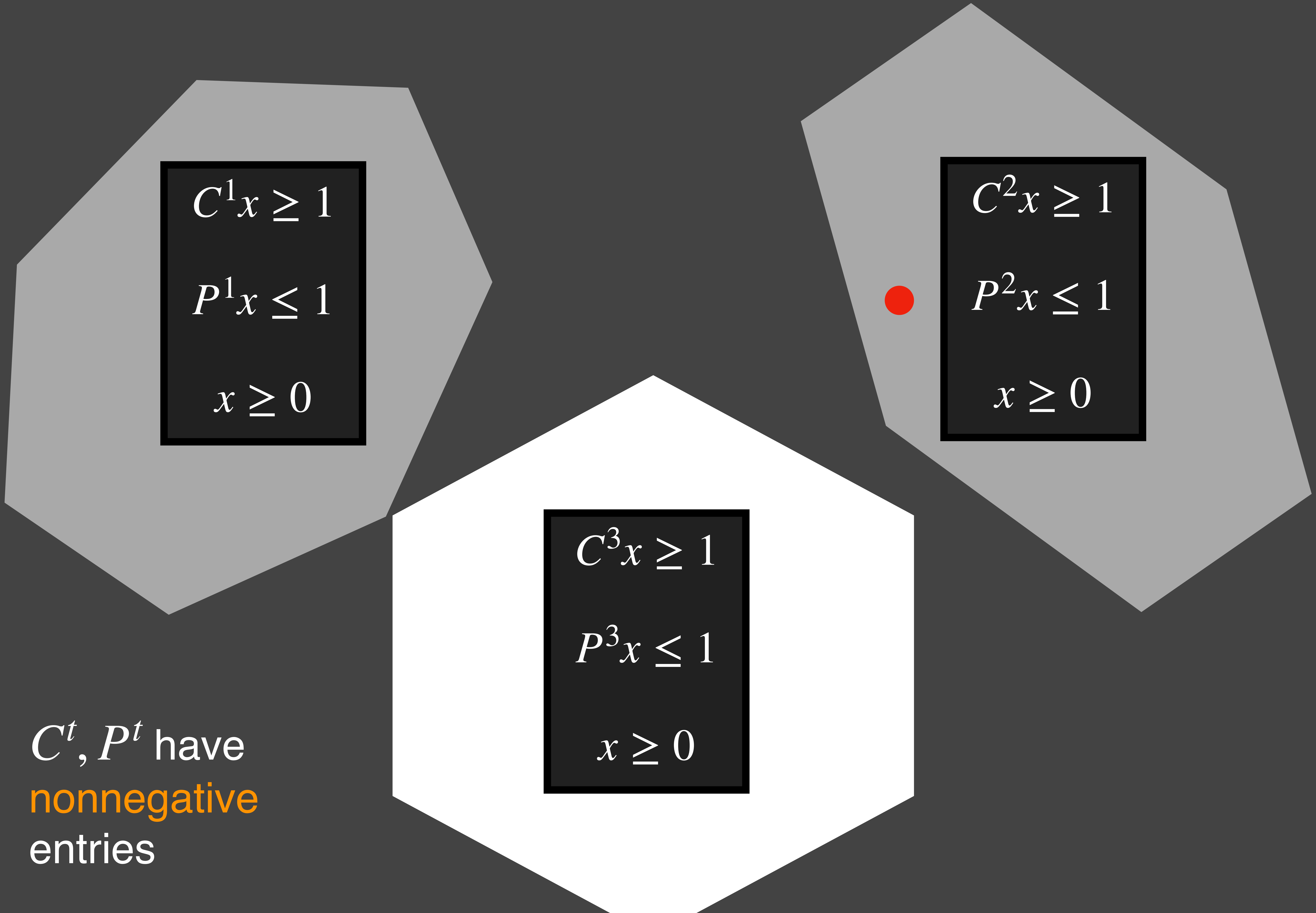

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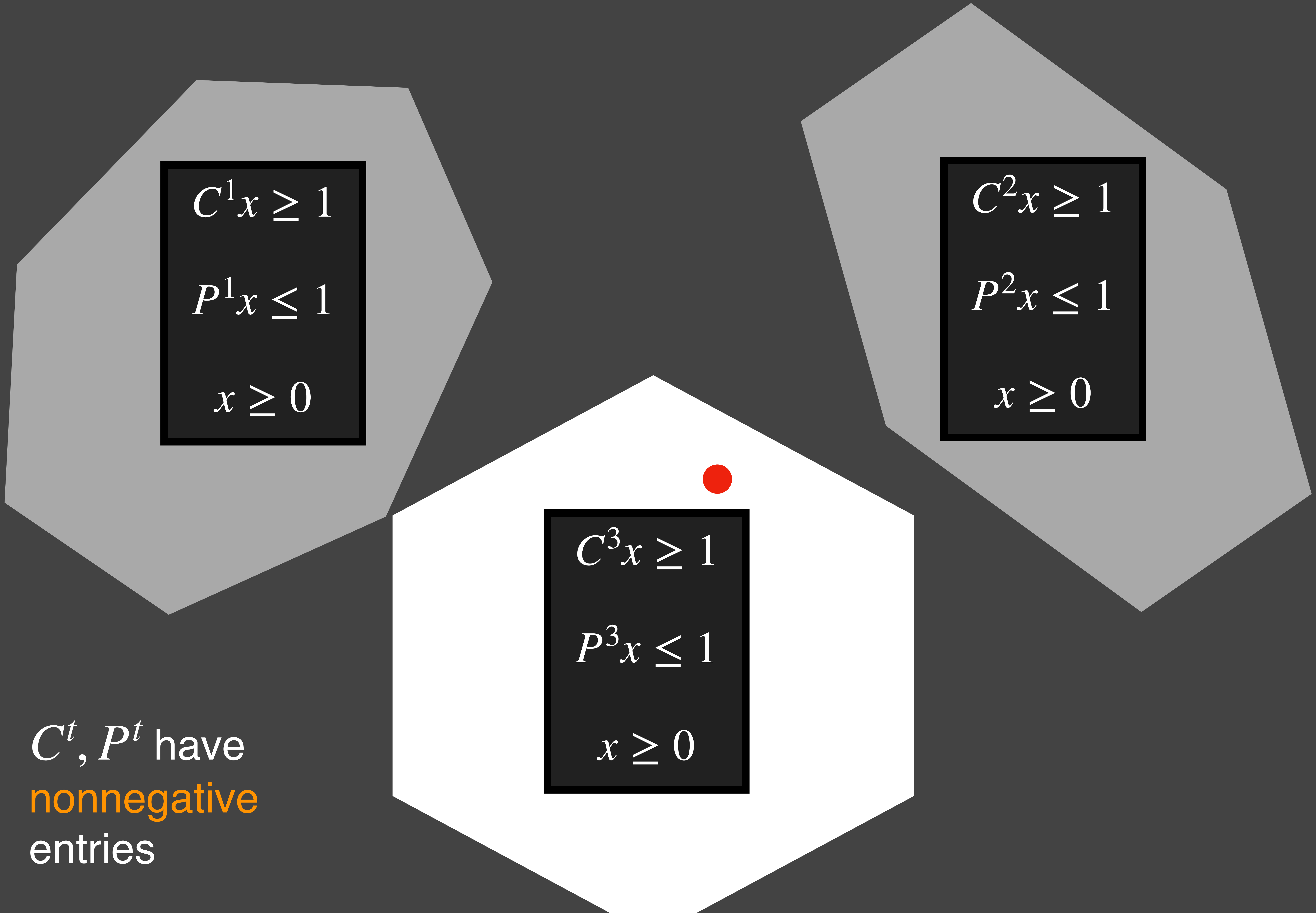

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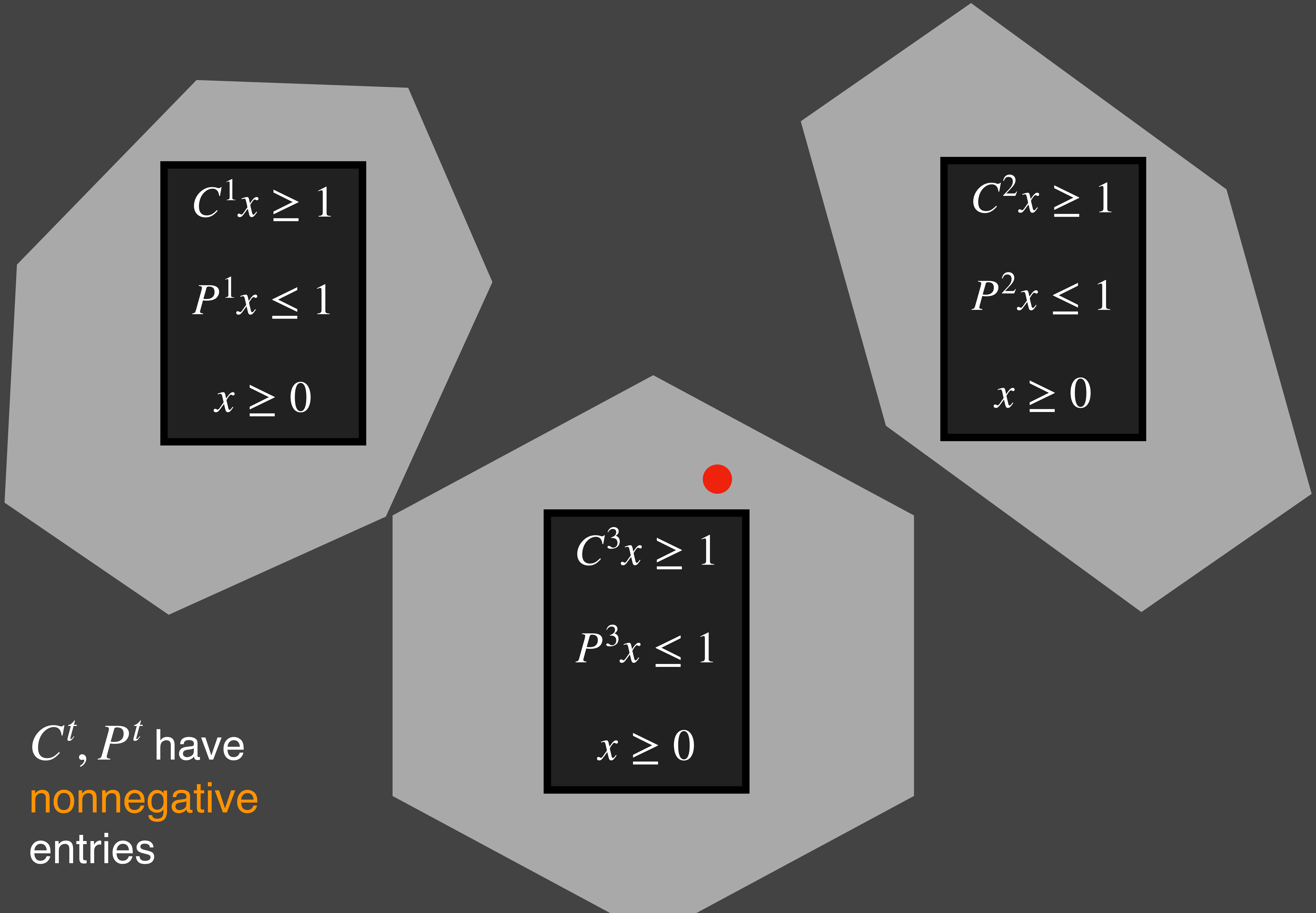

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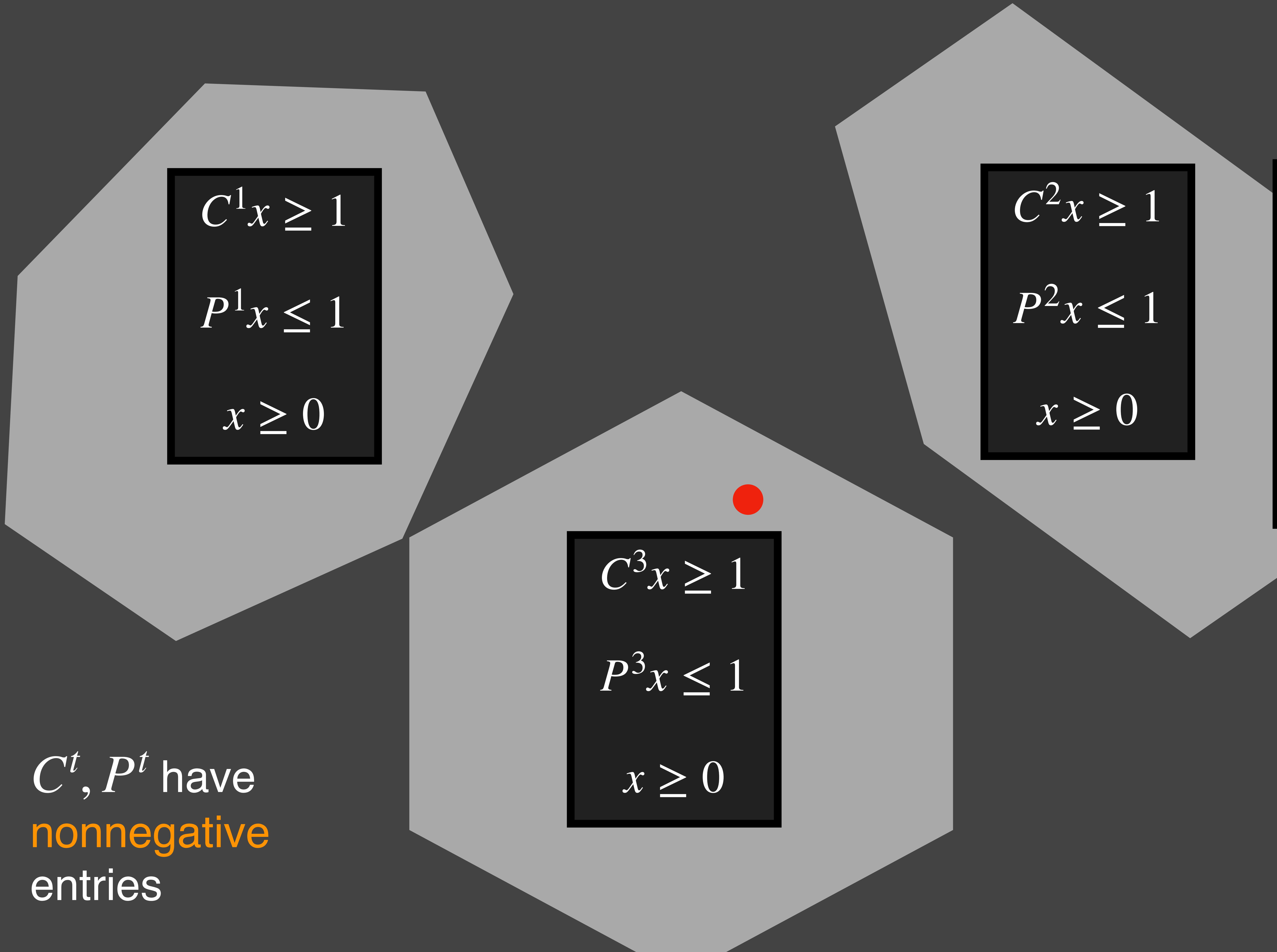

<u>Theorem</u> [Bhattacharya, Buchbinder, L., Saranurak]:

Positive Body Chasing with movement $O_\epsilon(\log n) \cdot$ OPT.*

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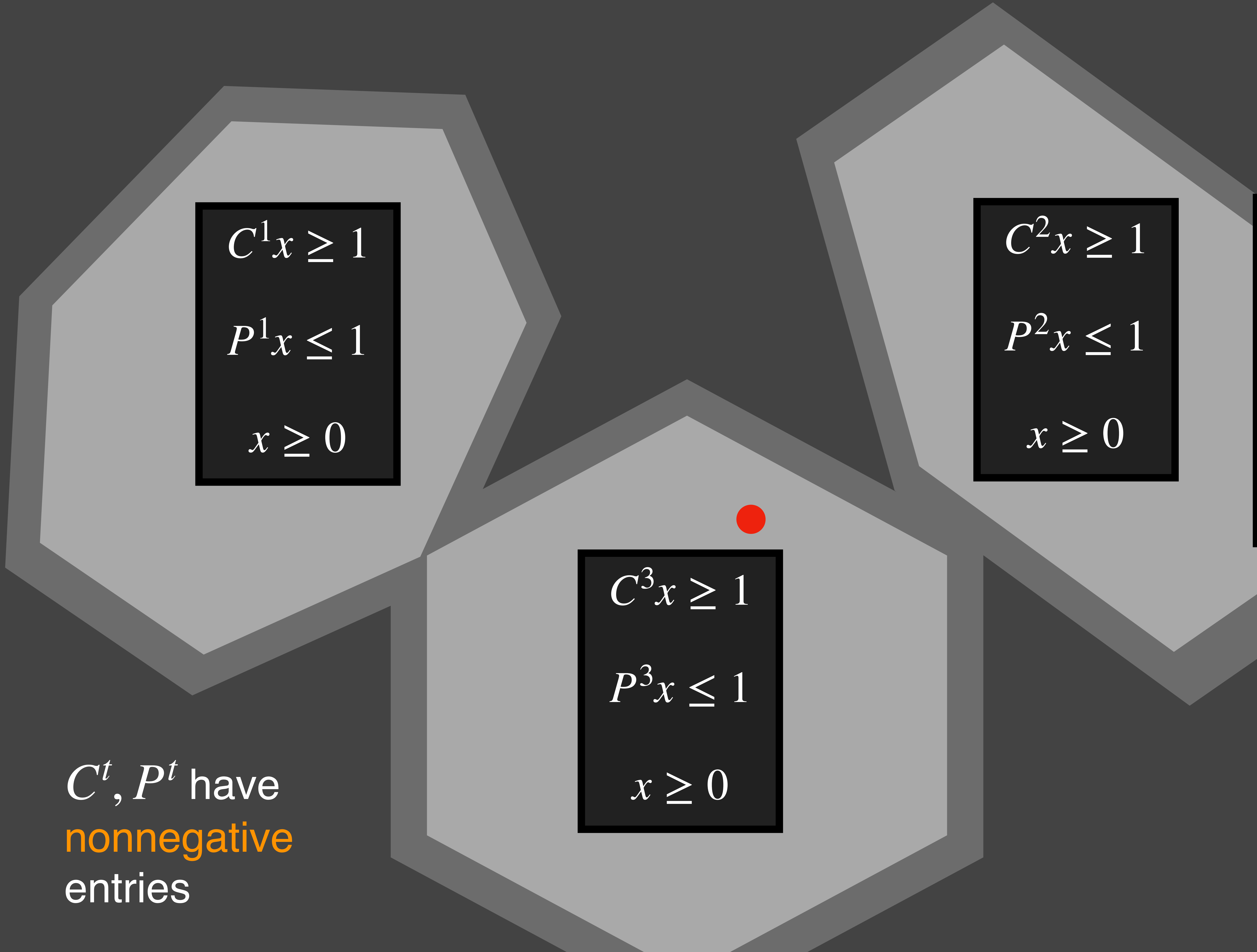

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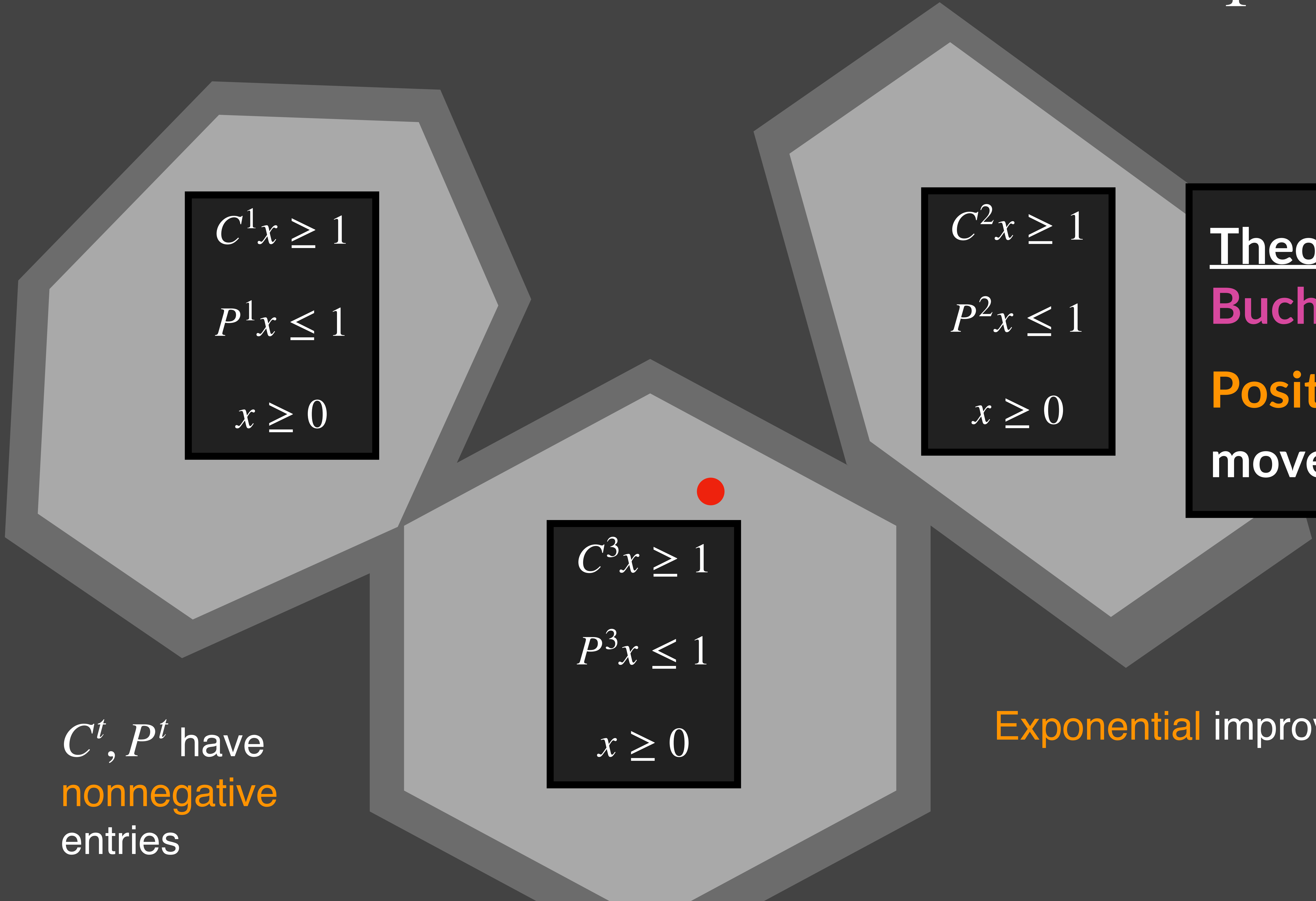

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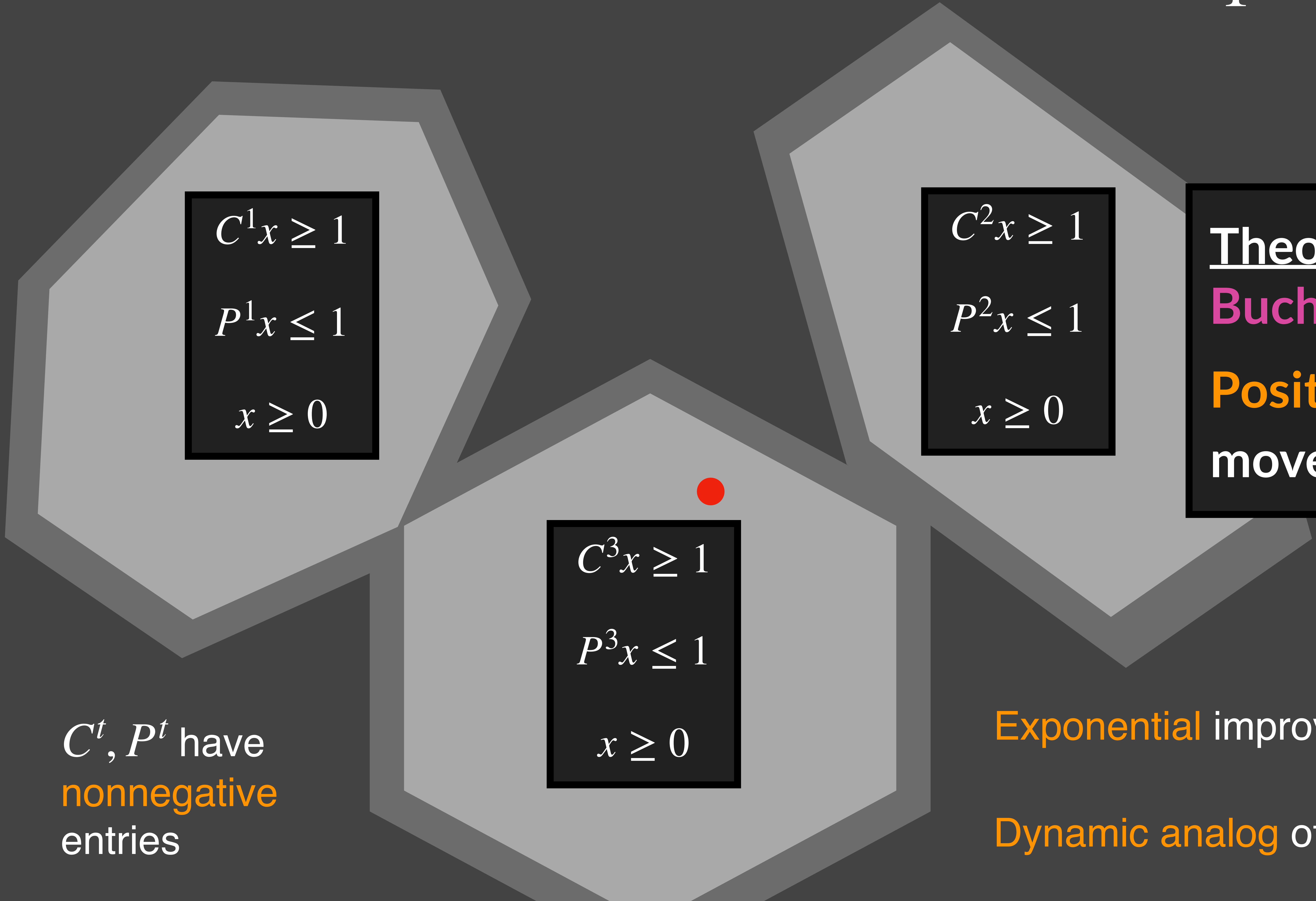

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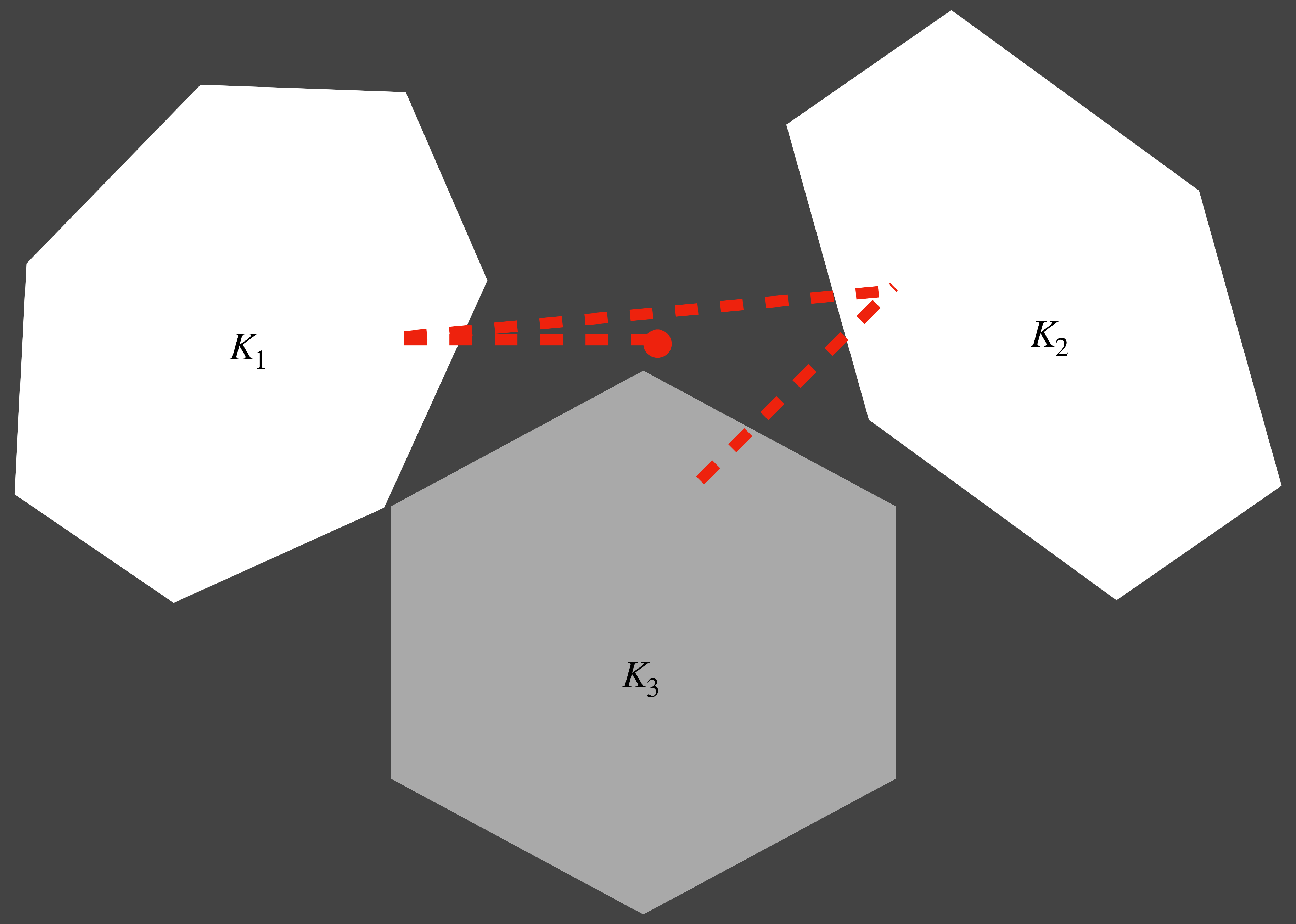
Dynamic analog of LP solver.

Absolute vs Competitive Recourse



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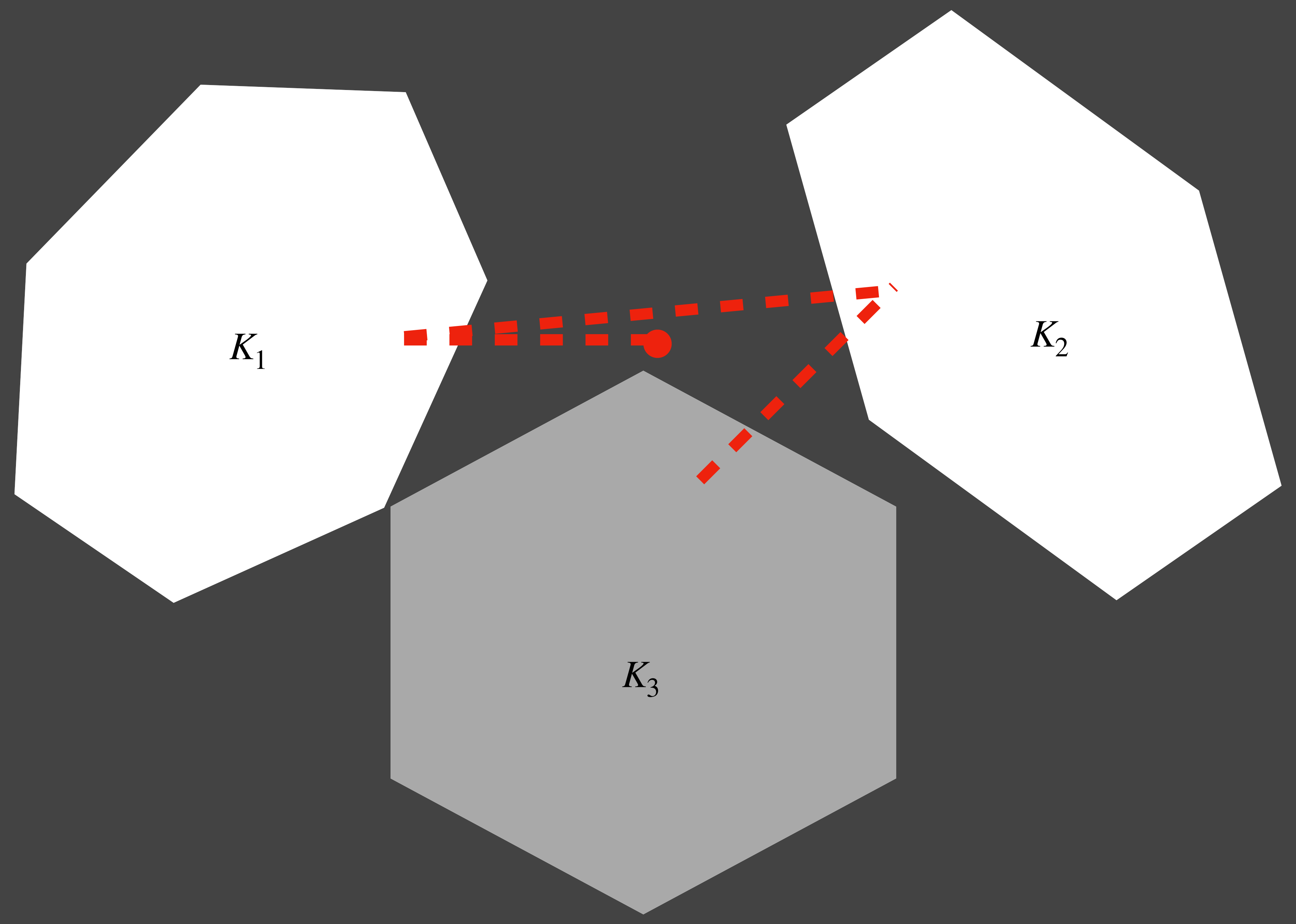

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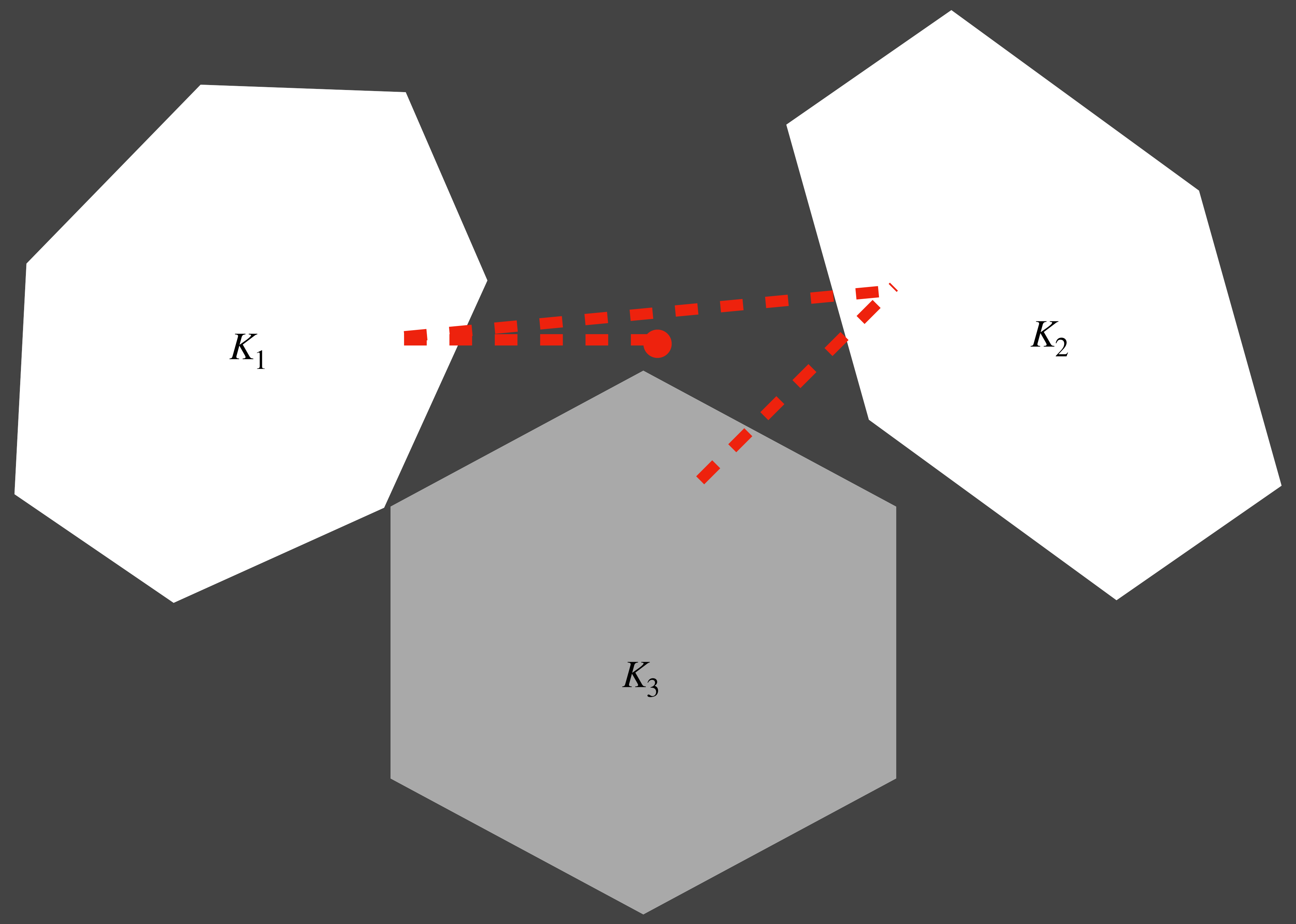

Absolute Recourse:

"Over T requests, algo has $\leq c \cdot T$ recourse."

Competitive Recourse:

"Over request sequence, algo has $\leq c \cdot$ OPT recourse."

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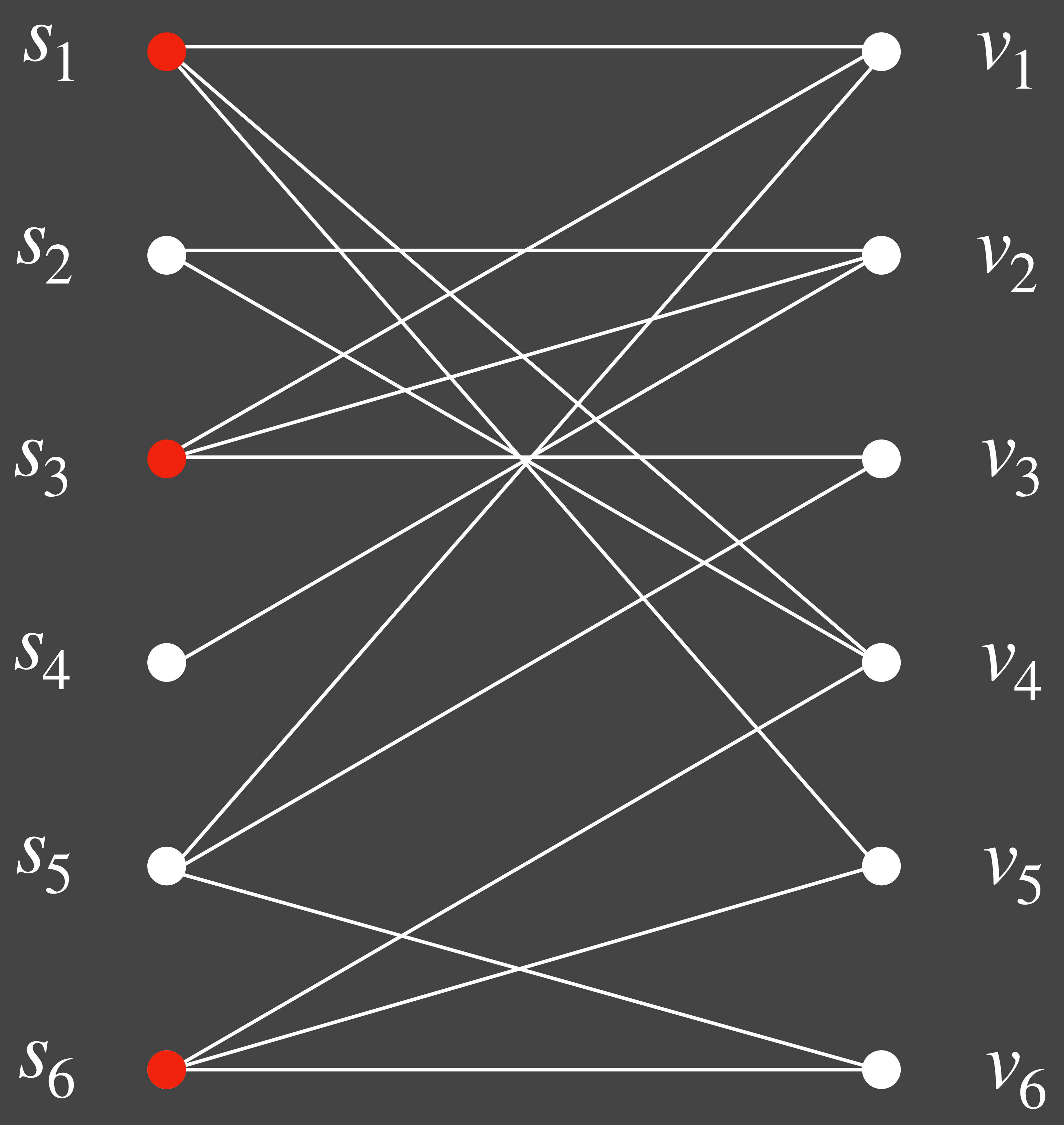

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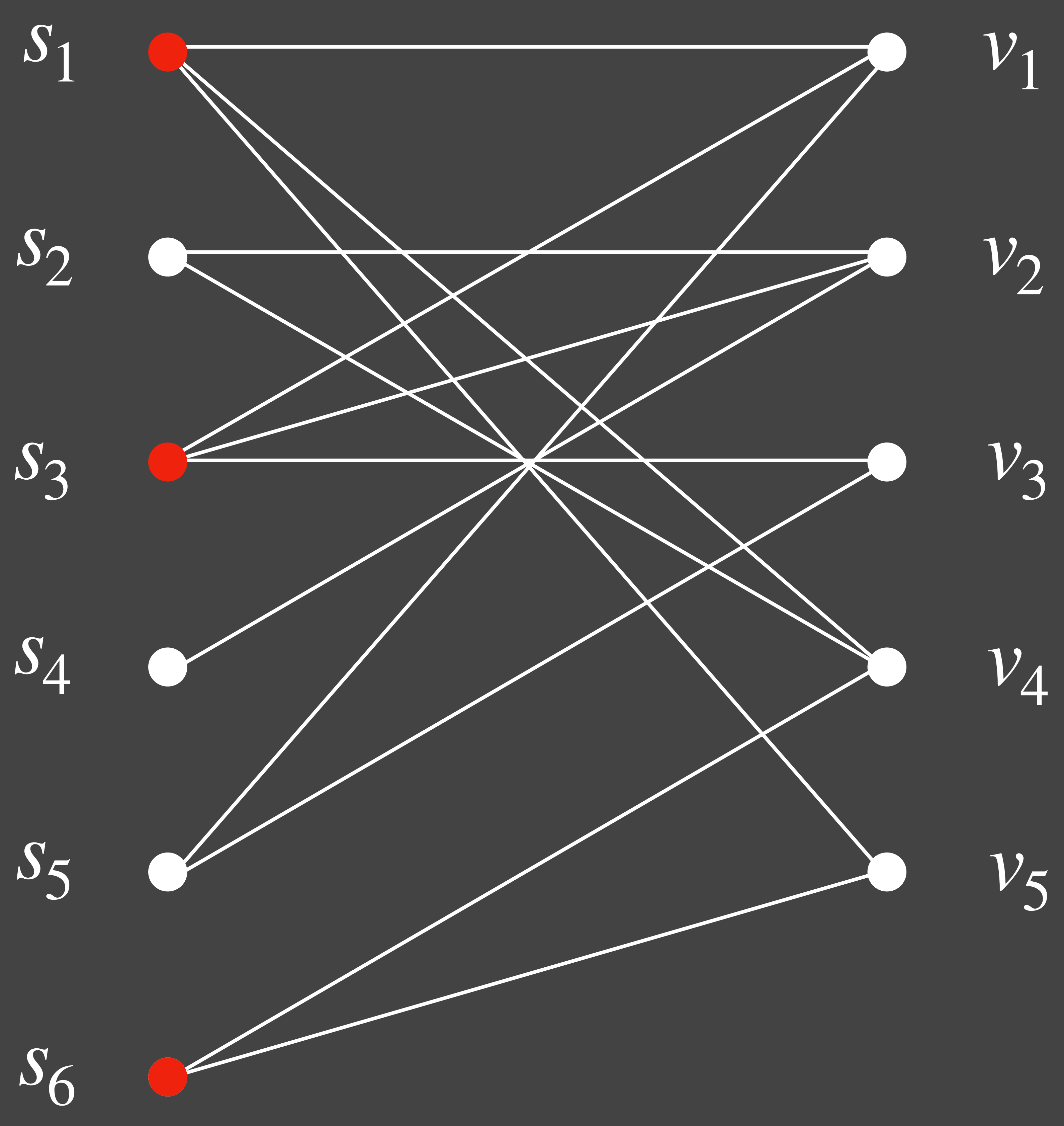

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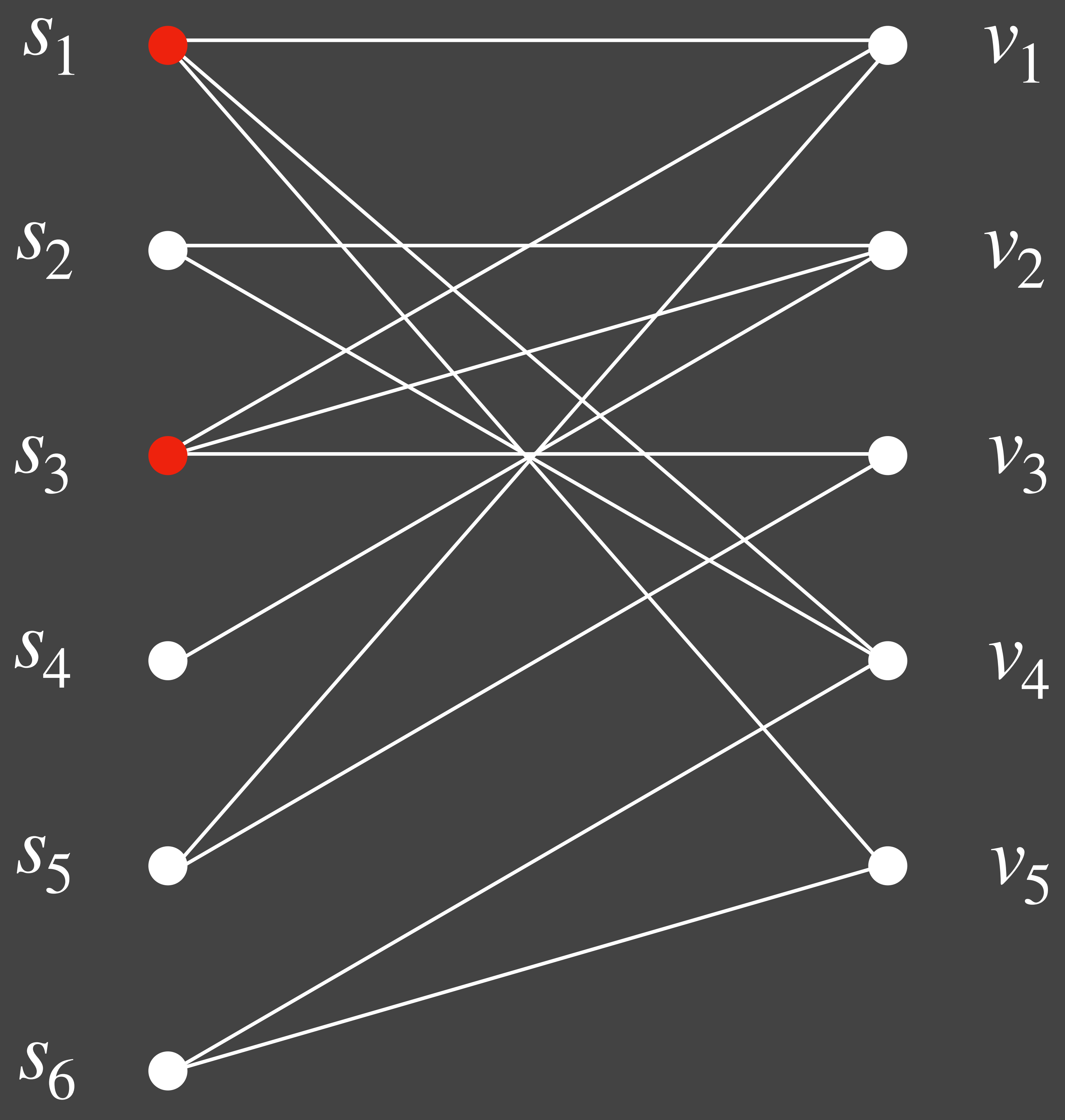

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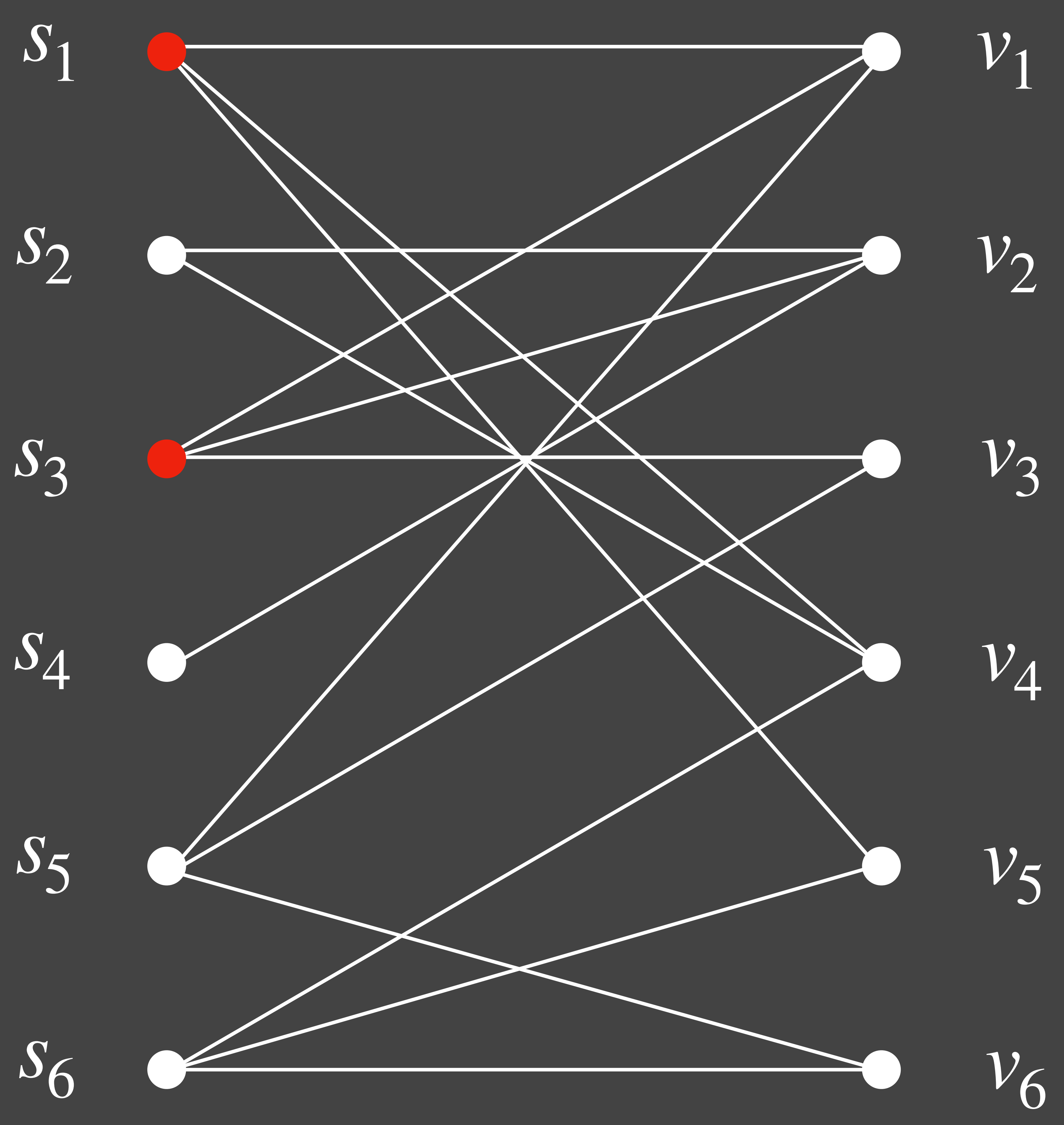

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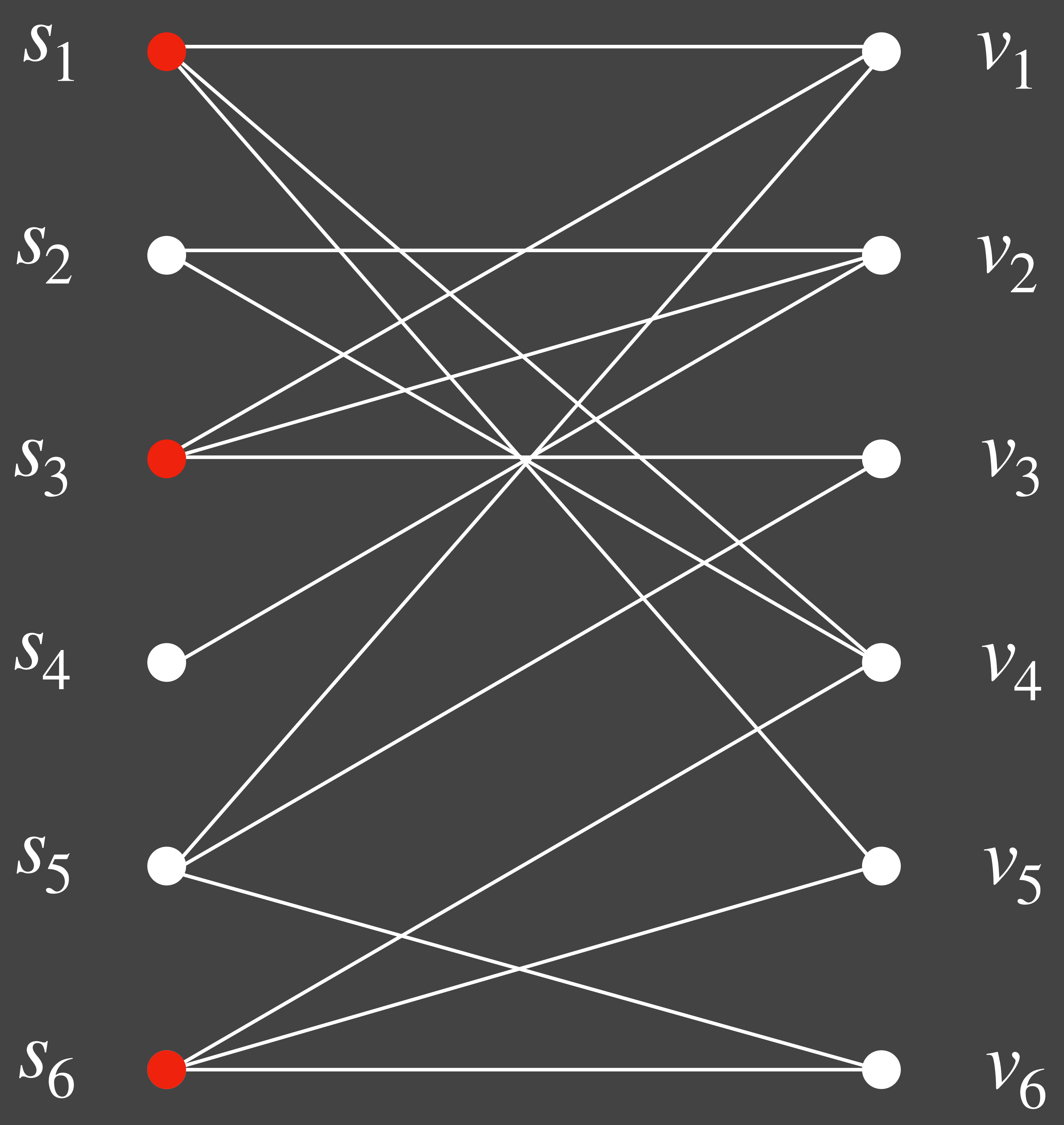

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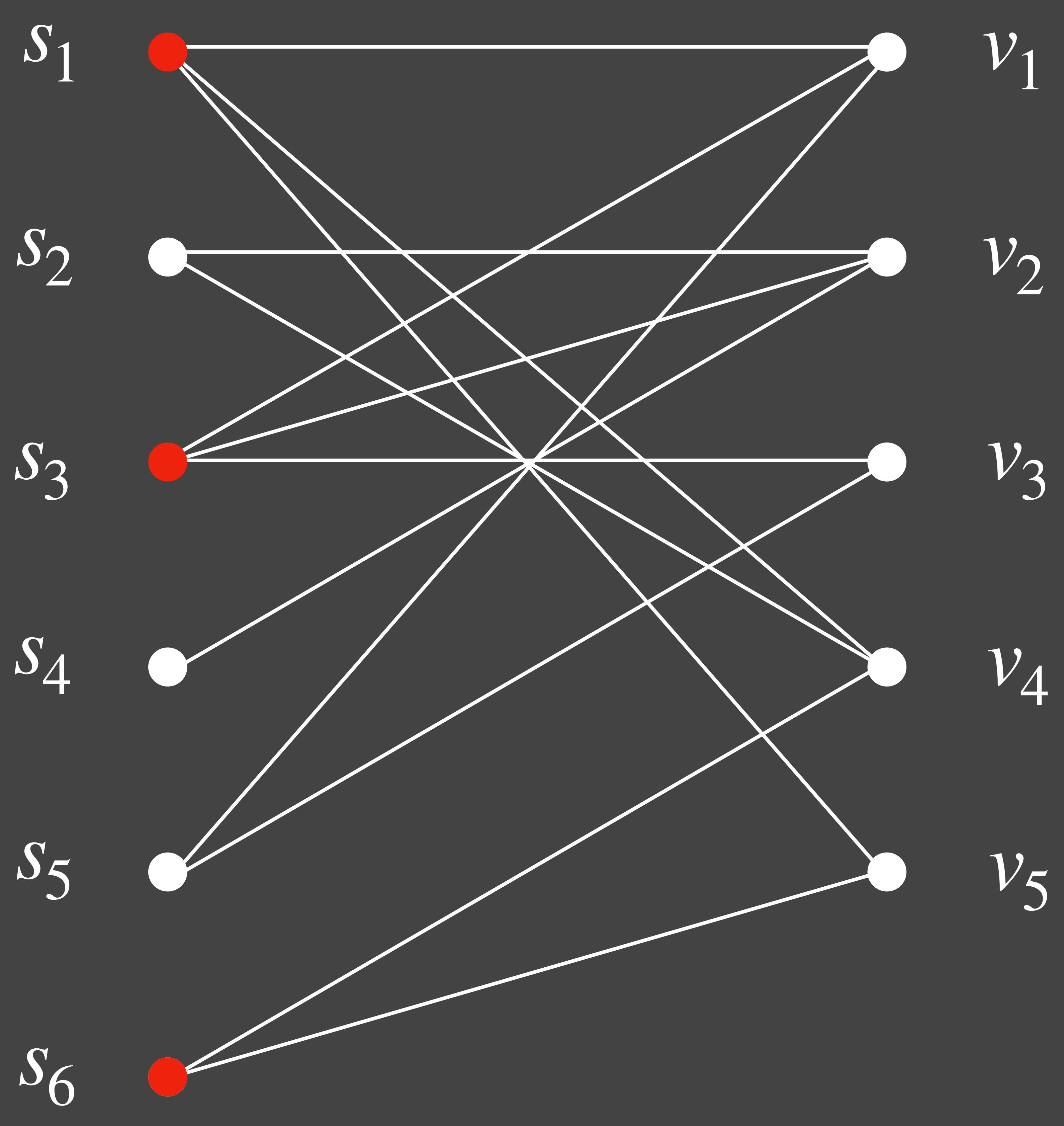

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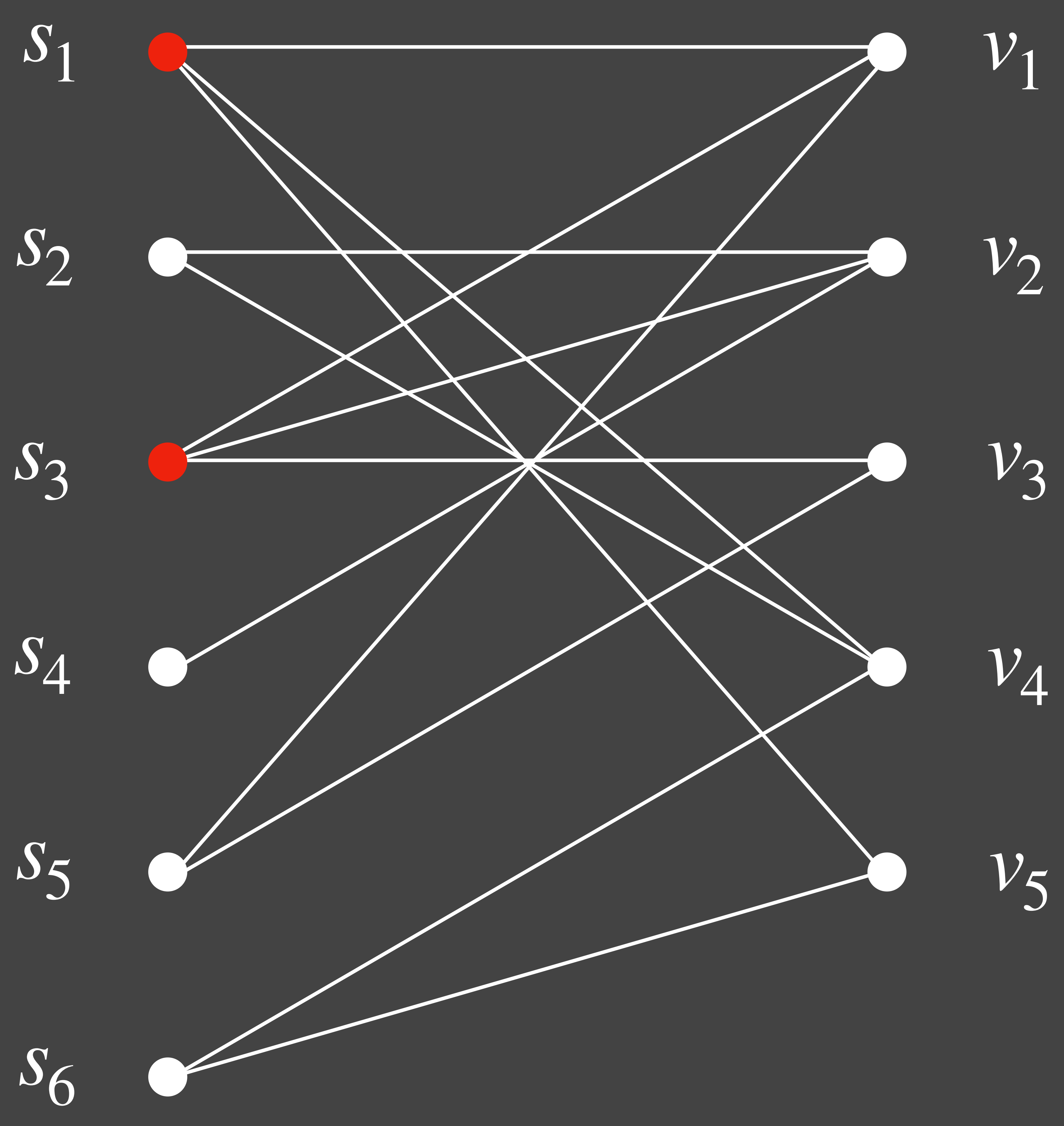

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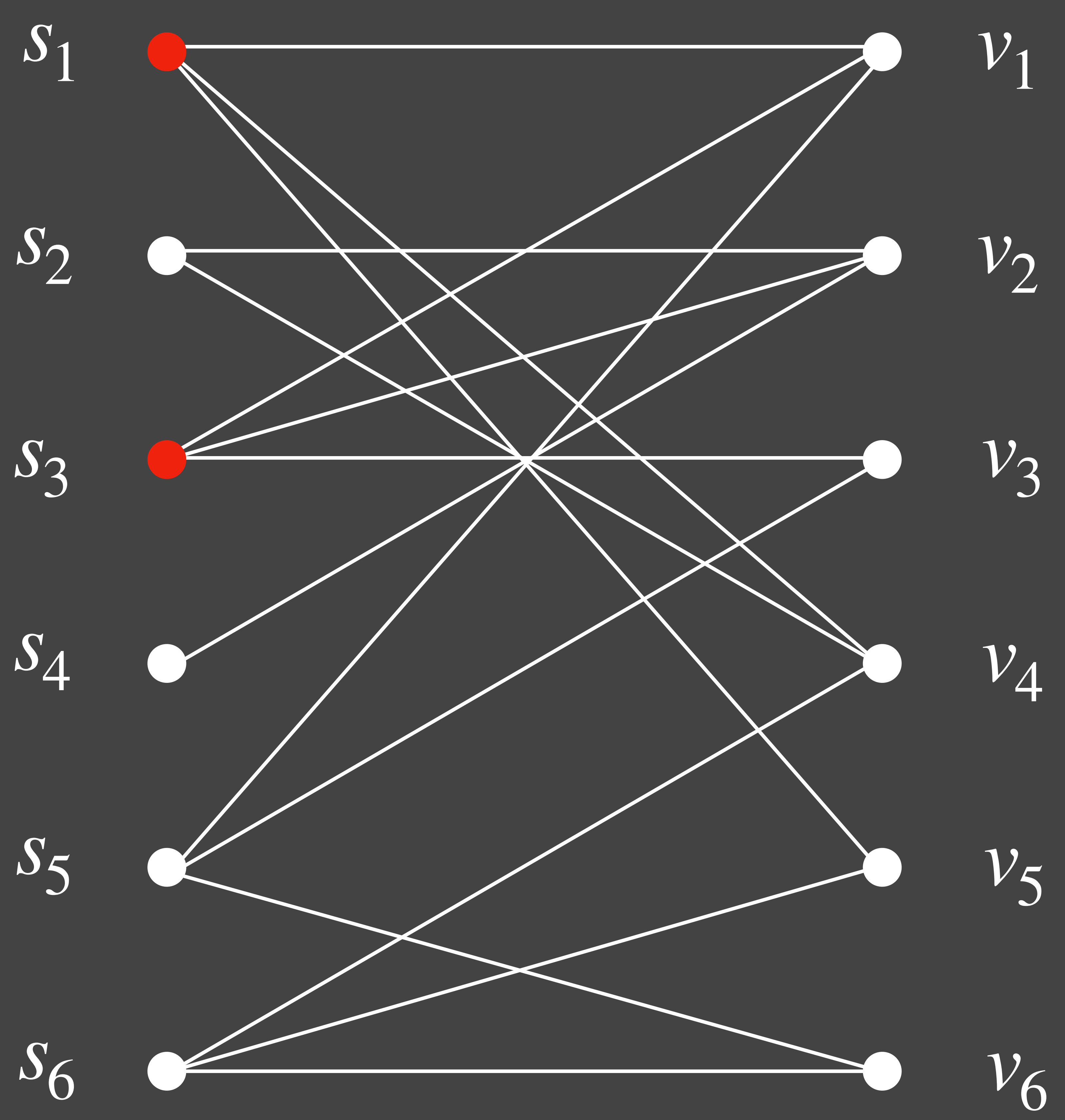

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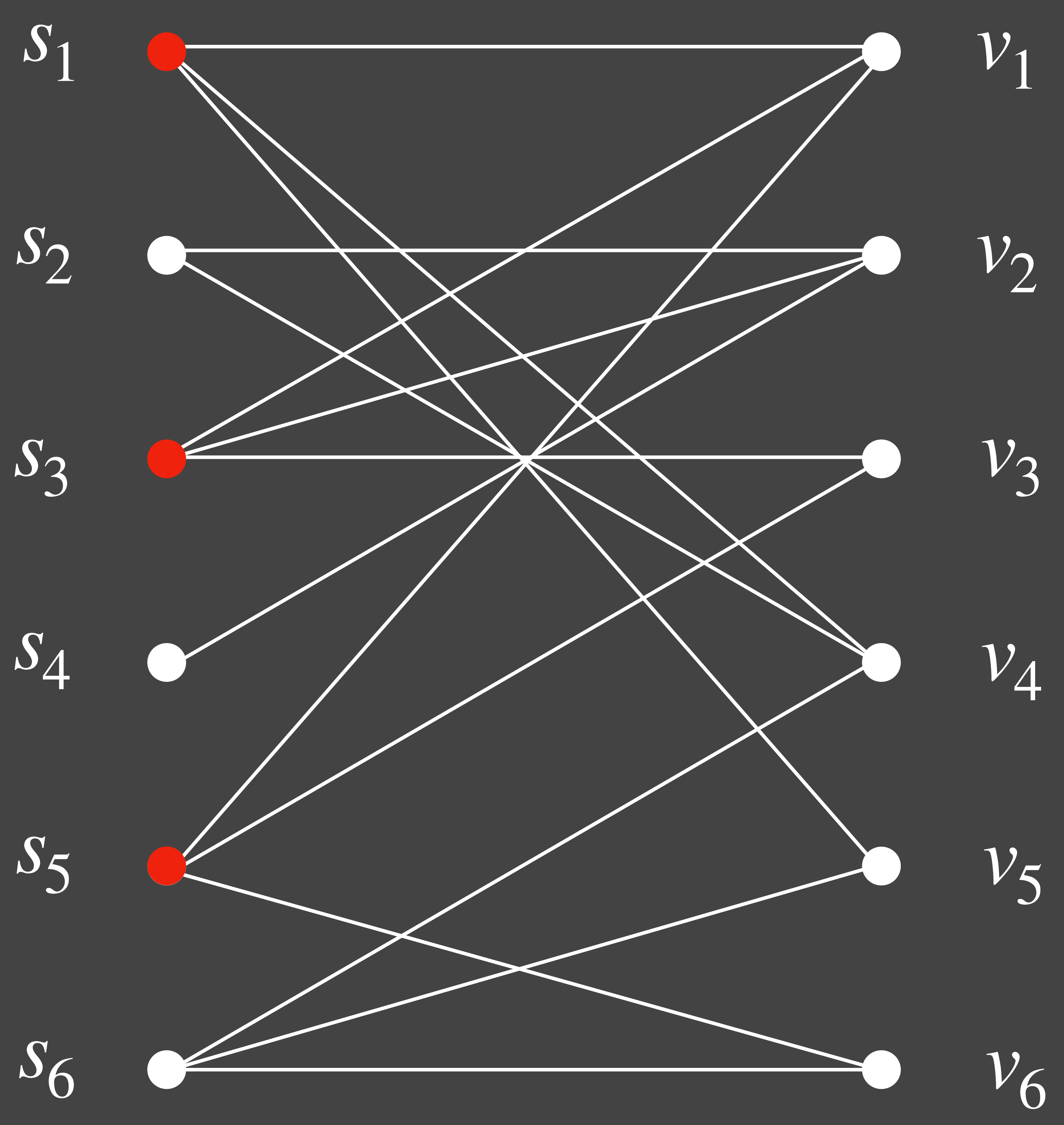

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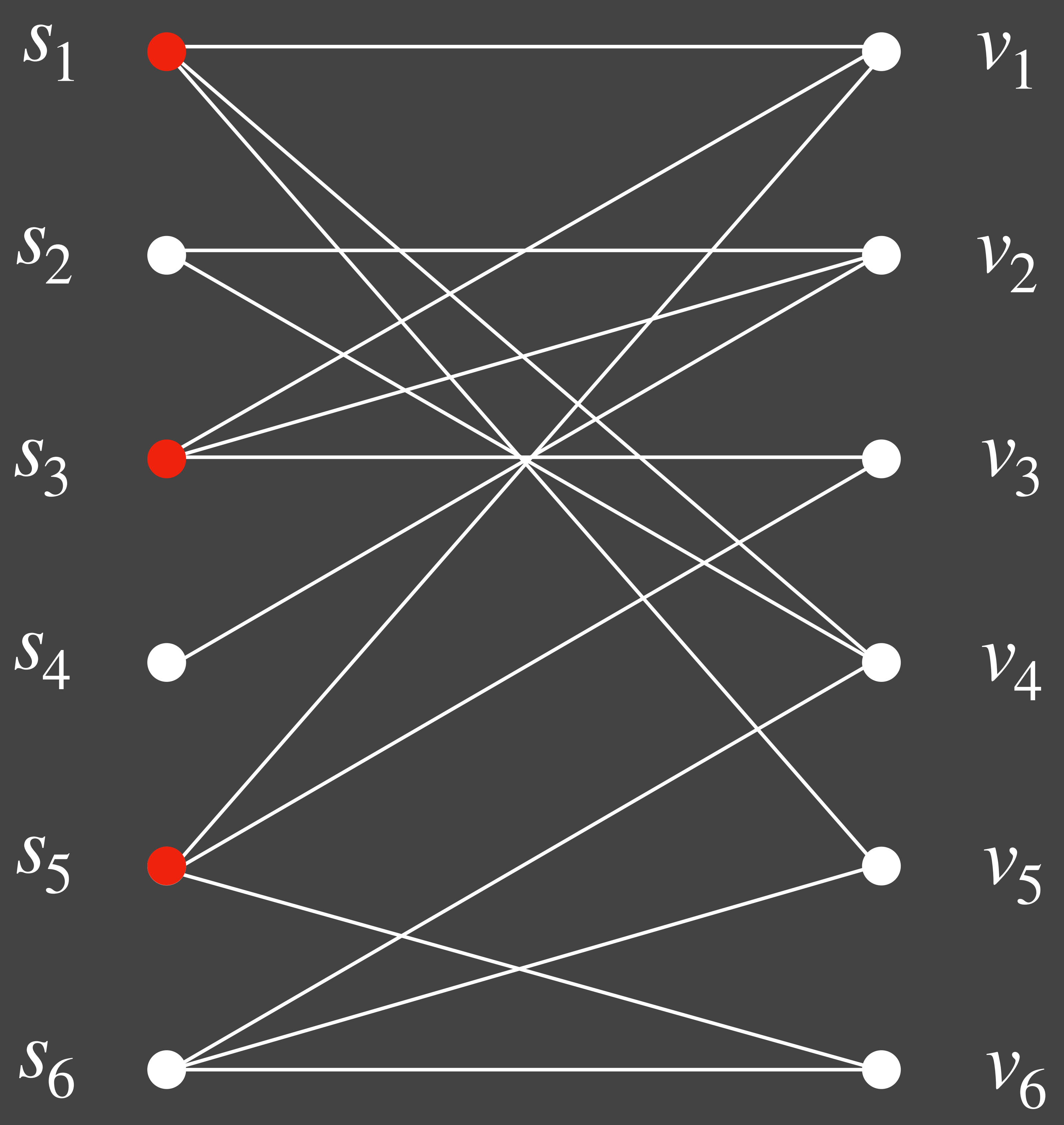

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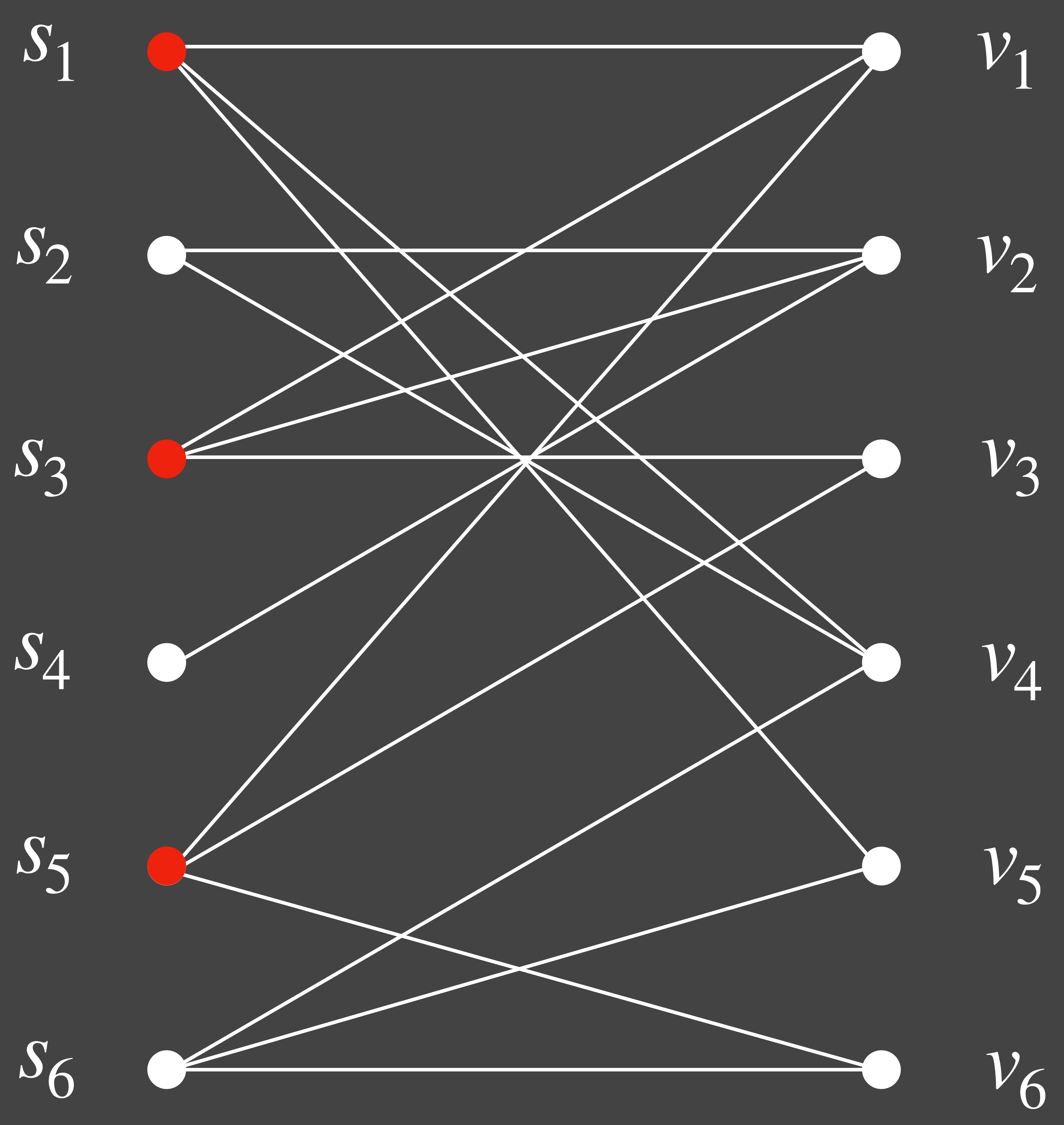

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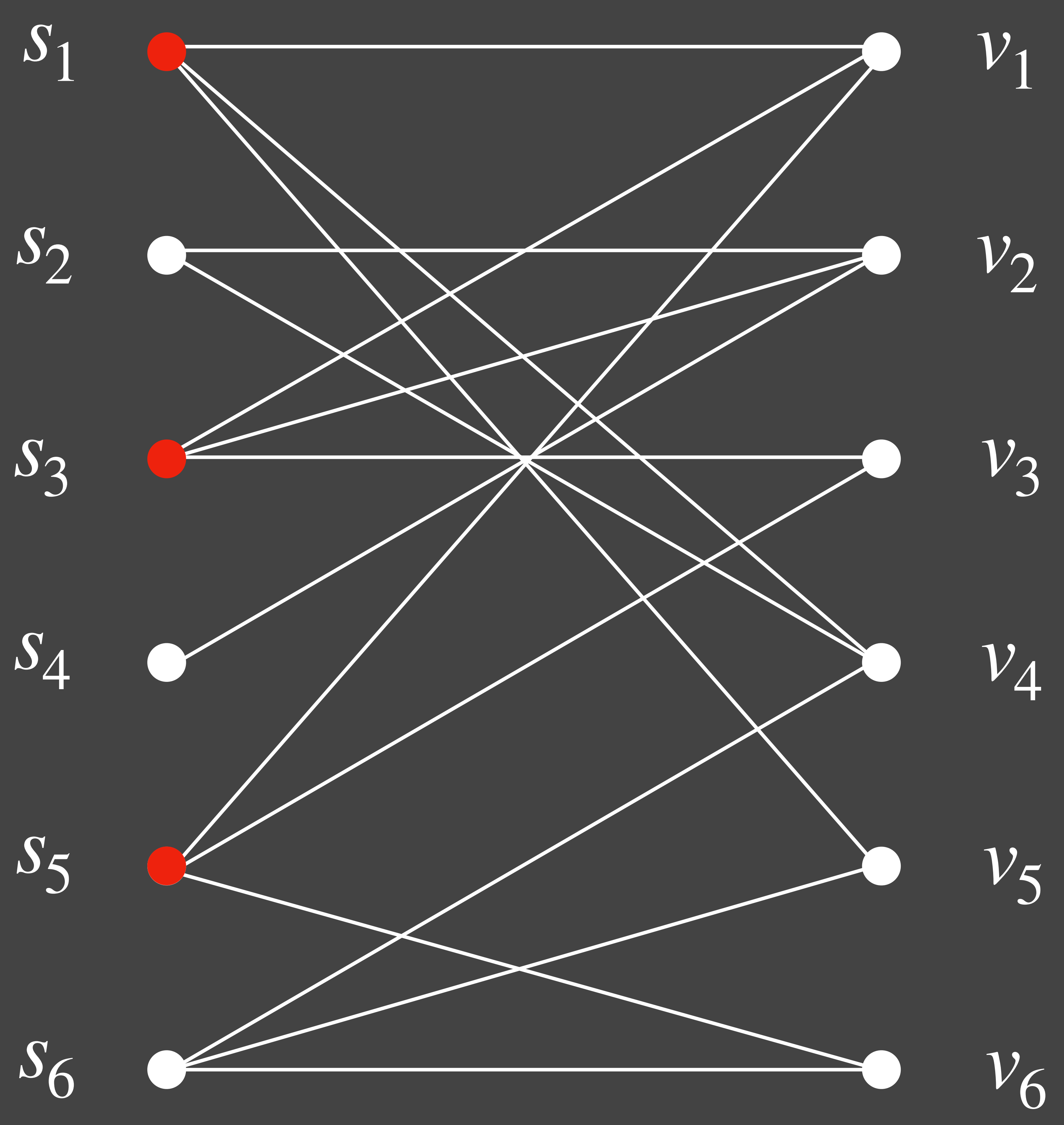

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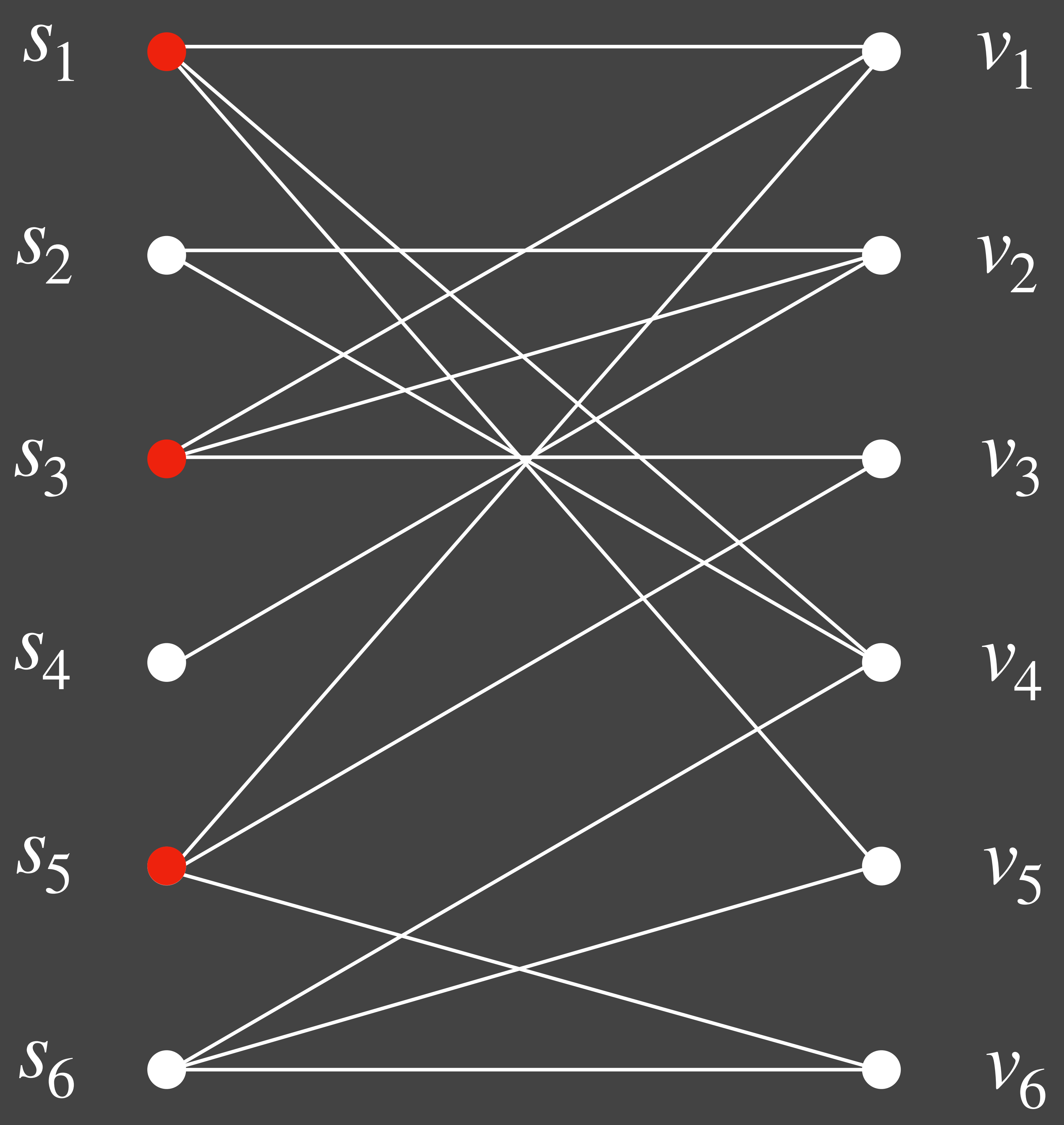

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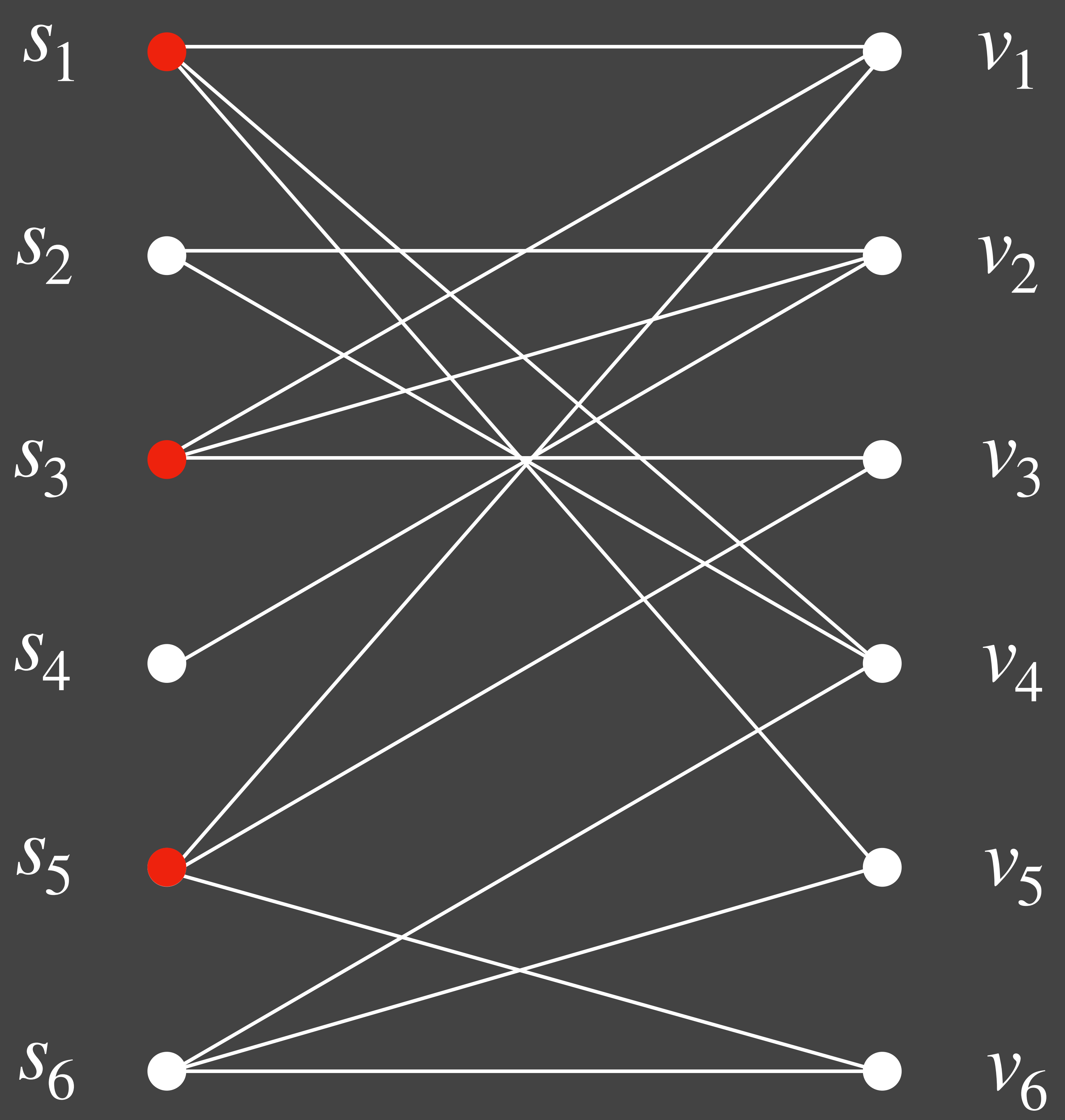

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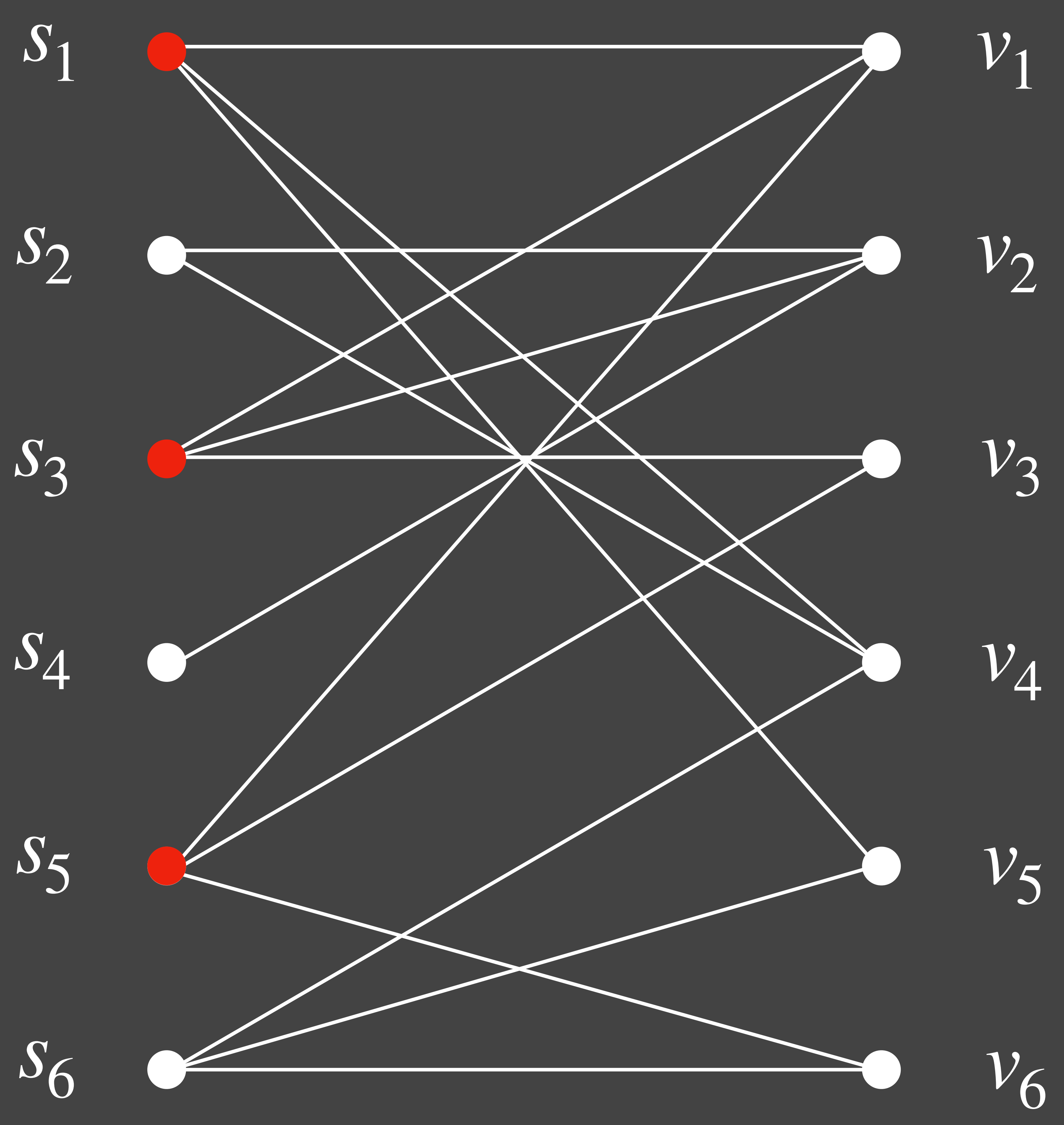

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Competitive recourse is the answer!

Applications via Rounding

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We show how to round fractional solutions to give:

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	Prior Work			Our Paper [BBLS]	
	Approx	Recourse	Ref	Approx	Recourse
Set Cover	O(log n)	O(T)	[GKKP 17]	O(log n)	O(log n log f) · OPT
	O(f)	O(T)		O(f)	O(f log f) · OPT
Load Balancing	2+ε	T · log n · poly(1/ε)	[KLS 23]	2+ε	poly(1/ε, log n) · OPT
Bipartite Matching	1+ε	O(T/ε)	[Folklore]	1+ε	poly(1/ε, log n) · OPT
Min. Spanning Tree	4	O(T)	[GK 14]	2+ε	poly(1/ε, log n) · OPT

Applications via Rounding

OPT := min recourse of any algorithm with same approx.

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Min. Spanning Tree	4	O(T)	[GK 14]	2+ε	poly(1/ε, log n) · OPT

Insert only.

The Fractional Algorithm

Step 0: Reduction to Chasing Halfspaces

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Claim: Suffices to give algorithm for chasing halfspaces.

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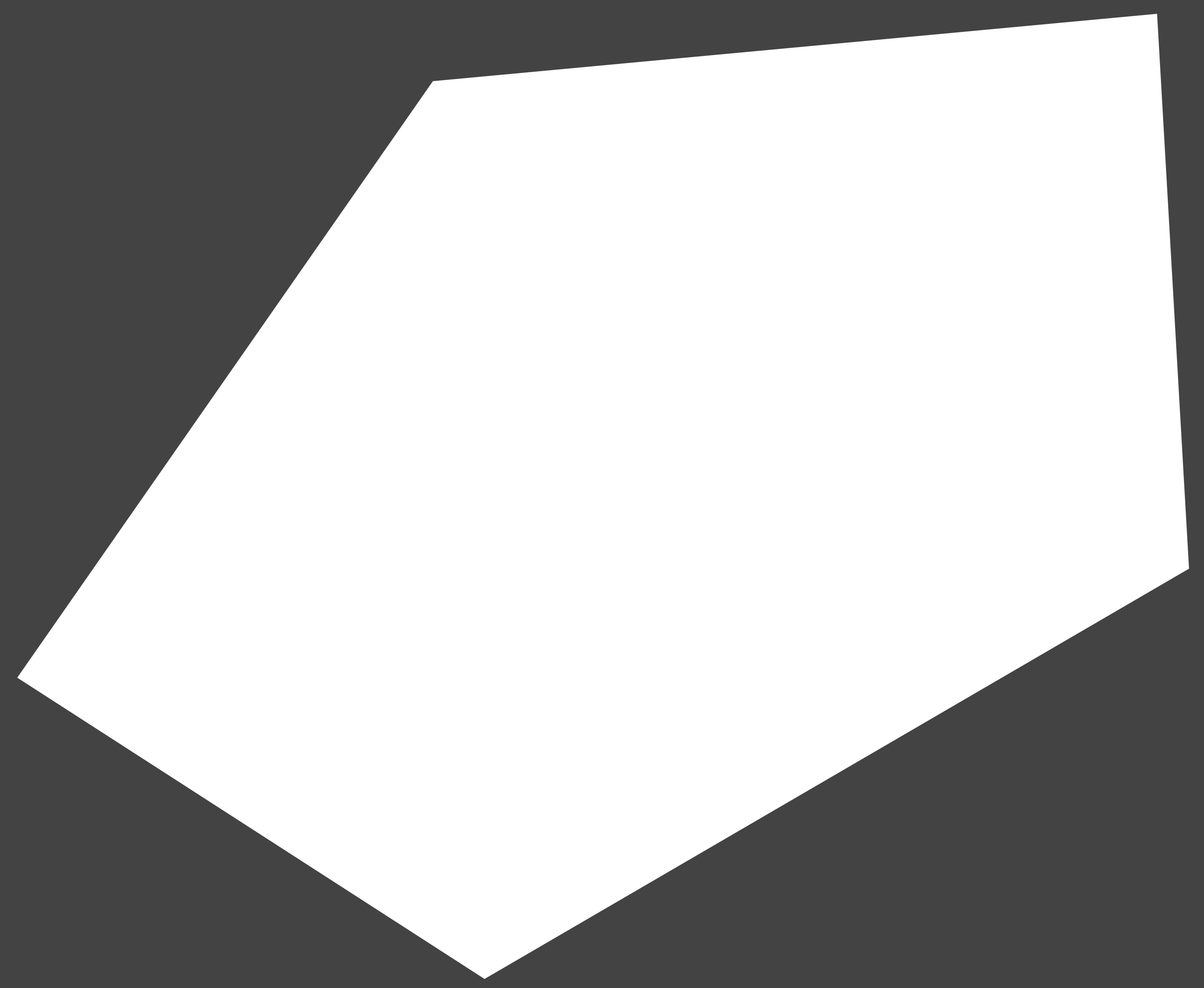

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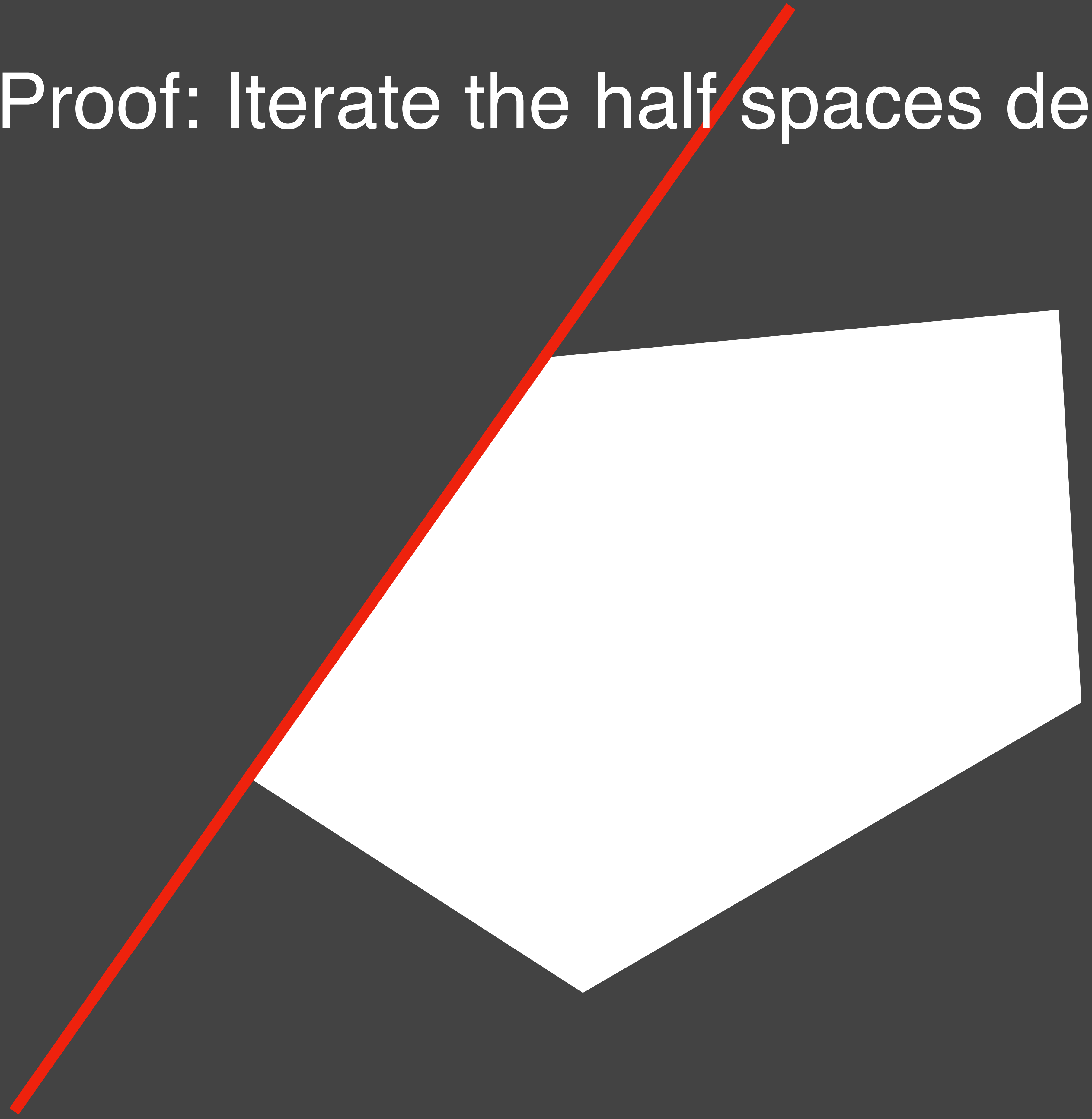

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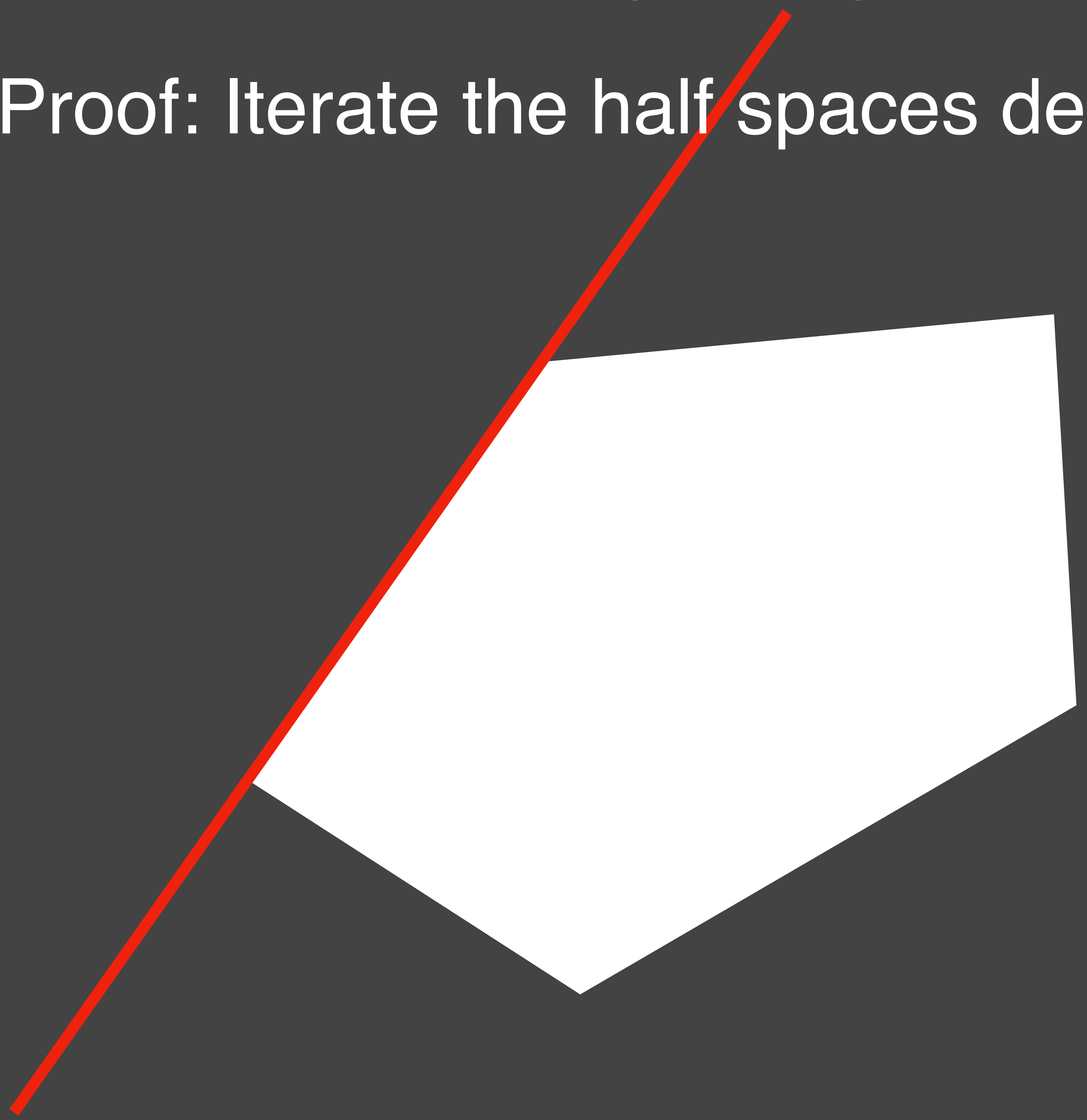

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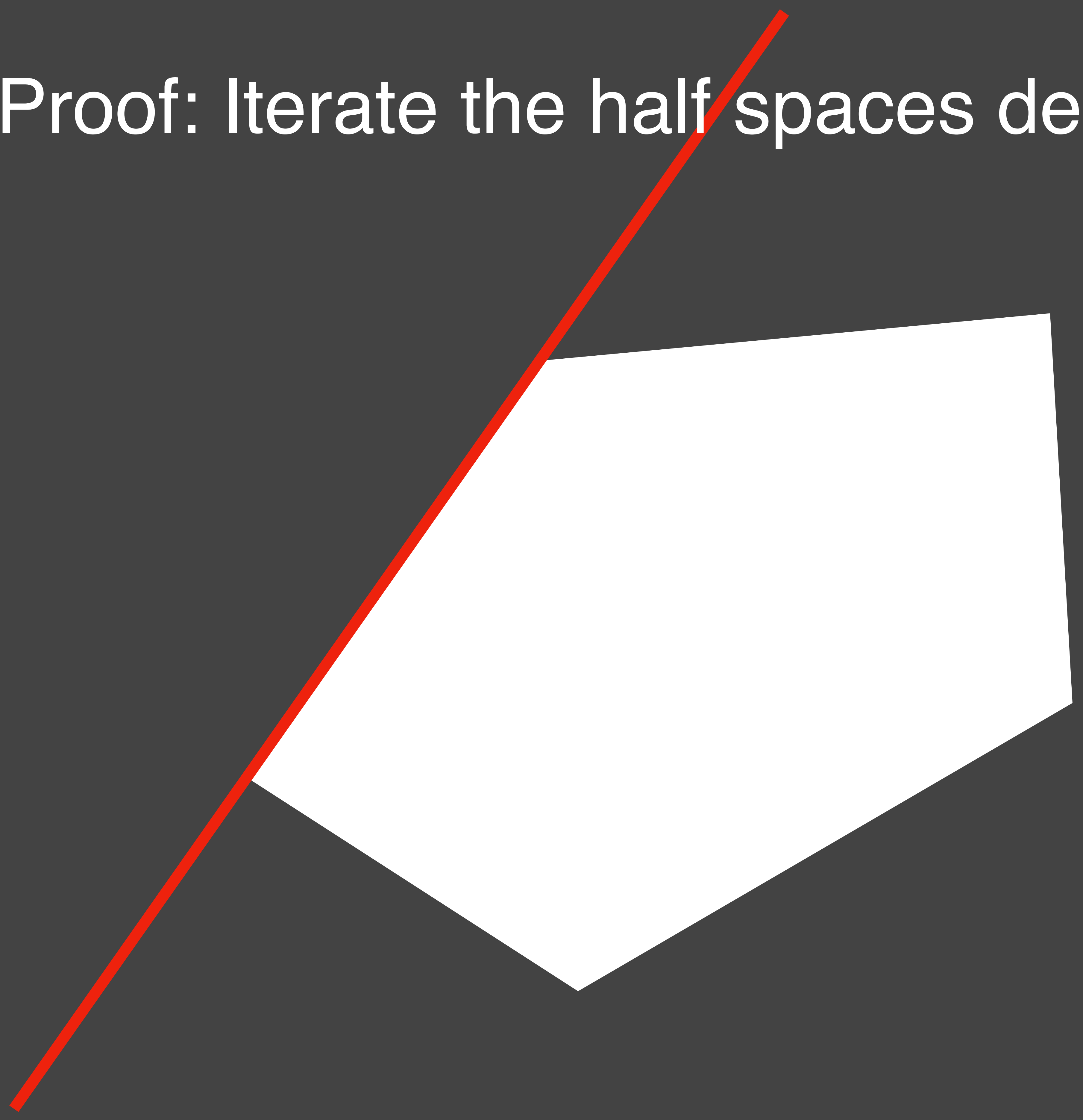

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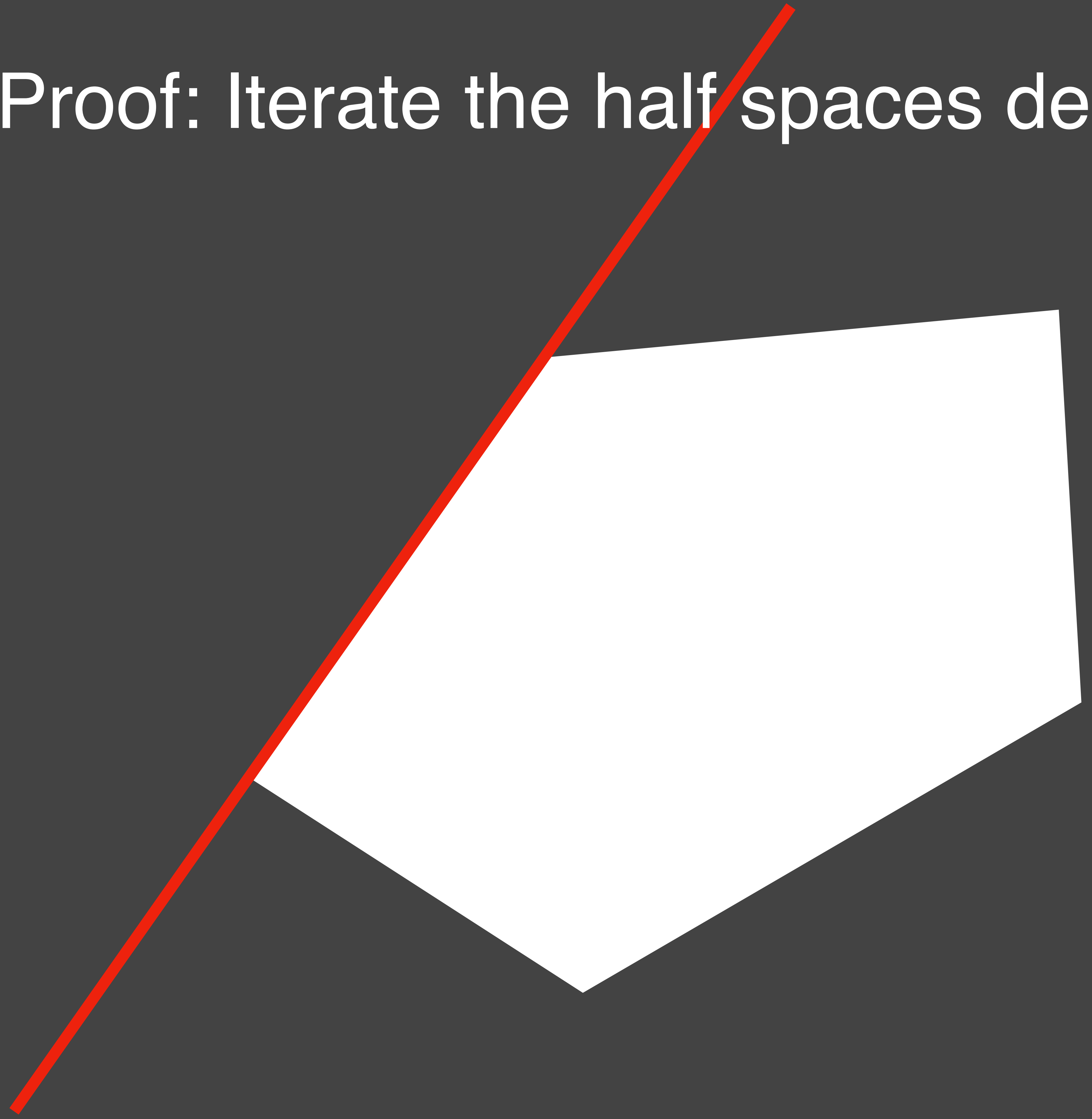

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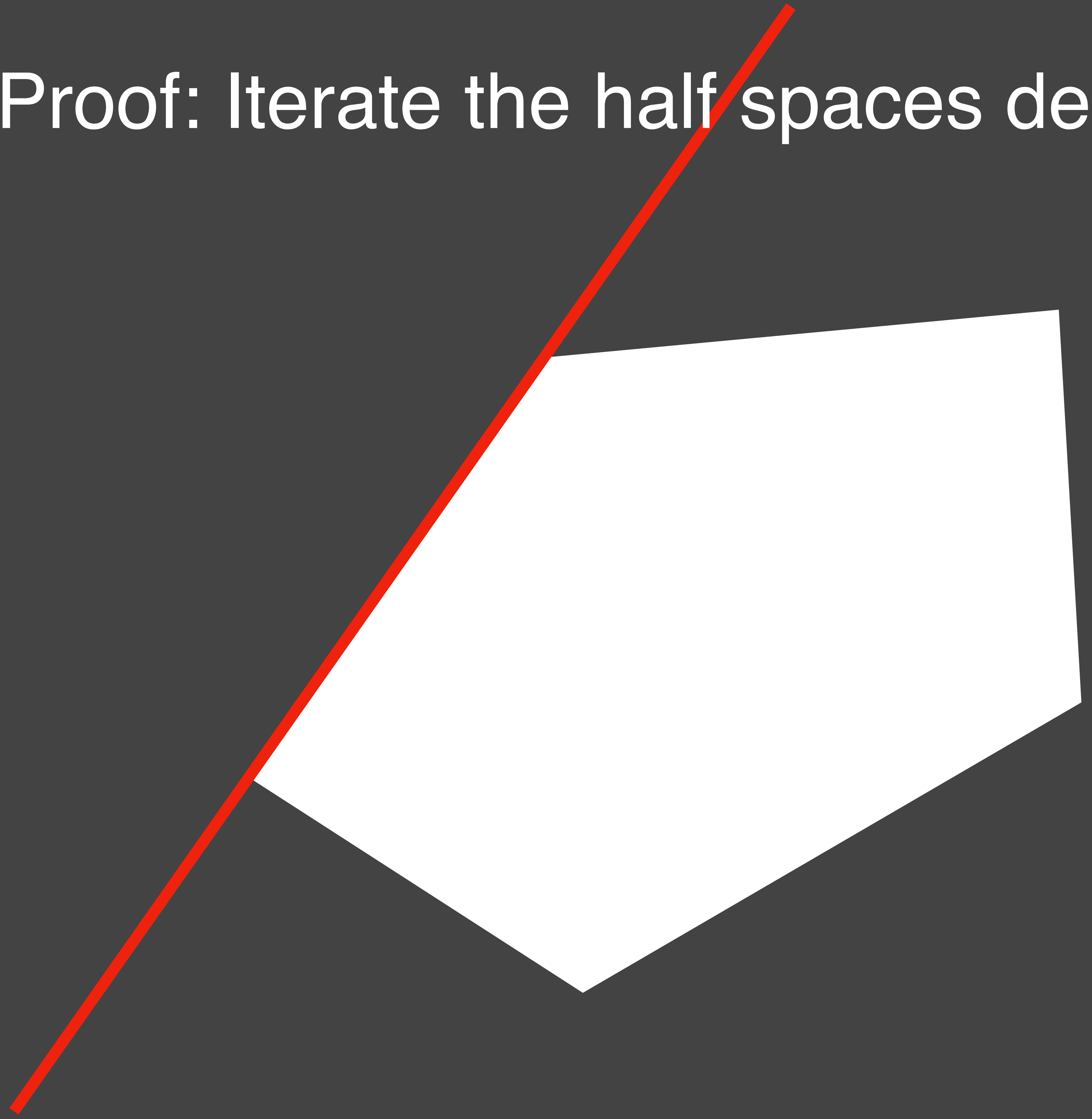

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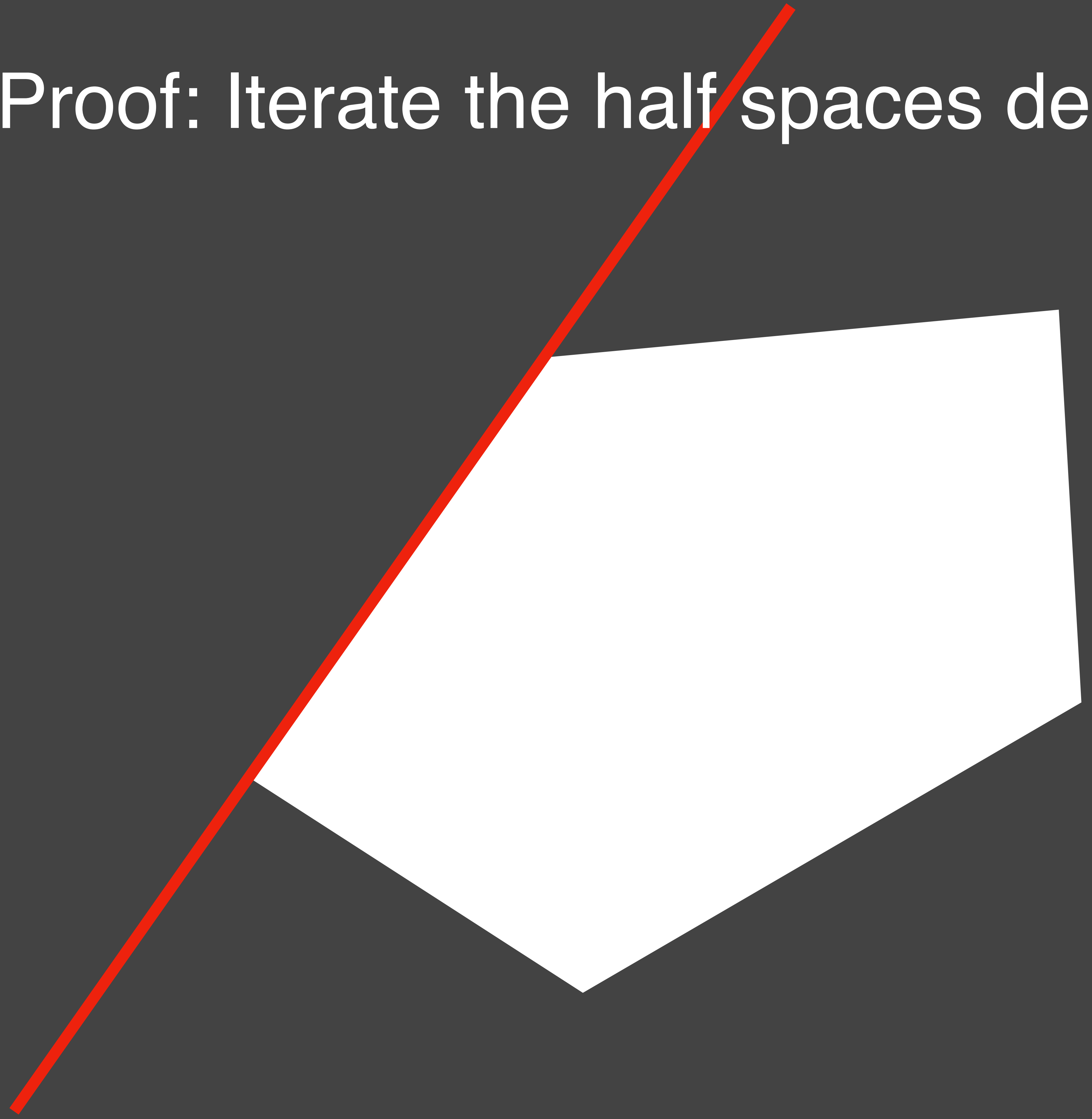

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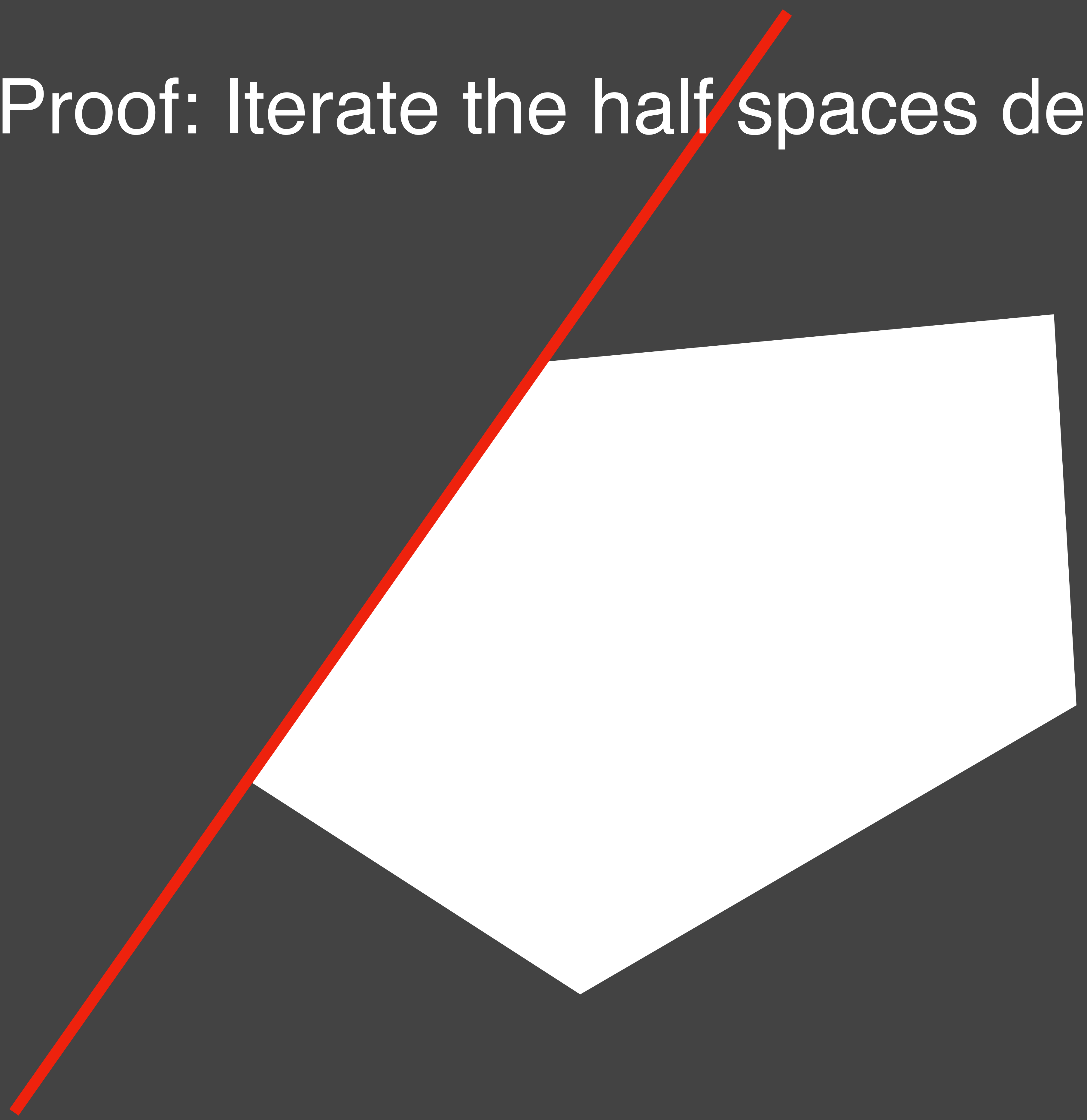

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I.e. all coefficients are positive, variables on same side of $\leq$.

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Multiplicative weights update (almost)!

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We fit a dual to ALG's solution! How?

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Theorem [BBLS]:

Positive Body Chasing with movement $\frac{1}{\epsilon}O\left(\log\left(\frac{n}{\epsilon}\right)\right) \cdot \text{OPT}$.

Relating ALG and Dual Objective

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Linear combination gives Lemma 2.

Slack for this argument needs resource augmentation, i.e. violate packing by ϵ.

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Online Covering [Buchbinder Naor 09]

Our KL Projection Algorithm

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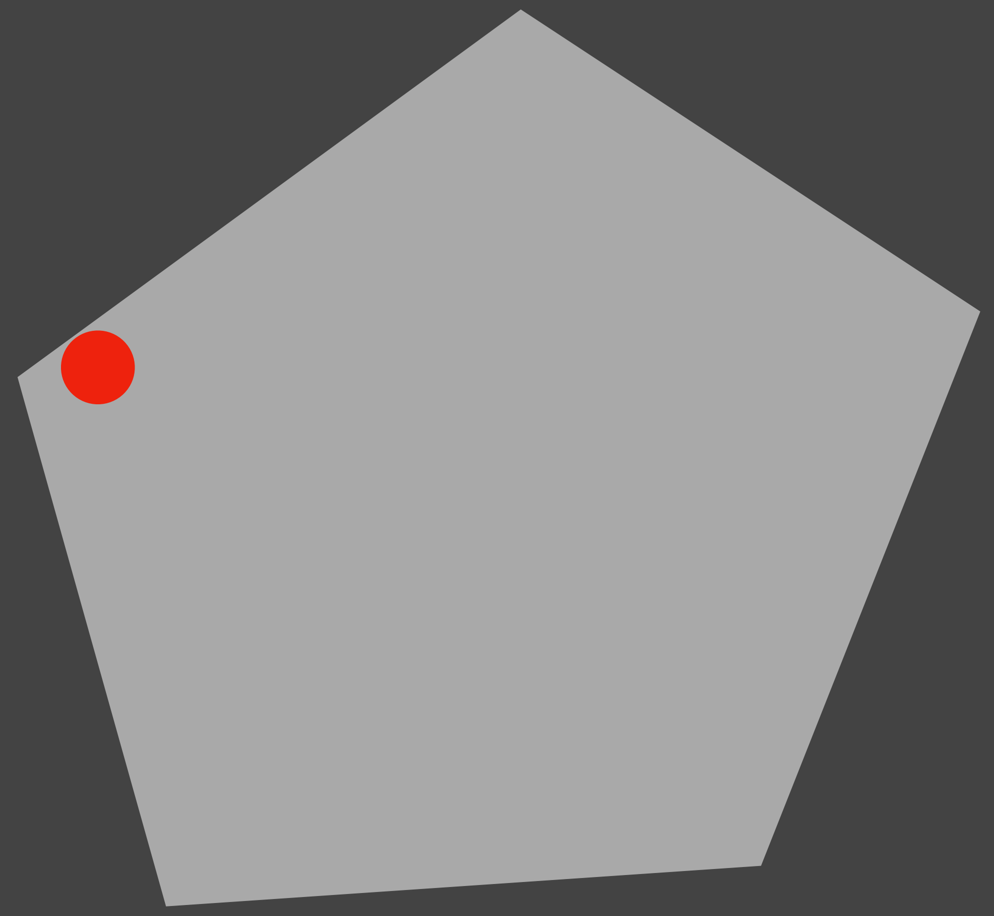

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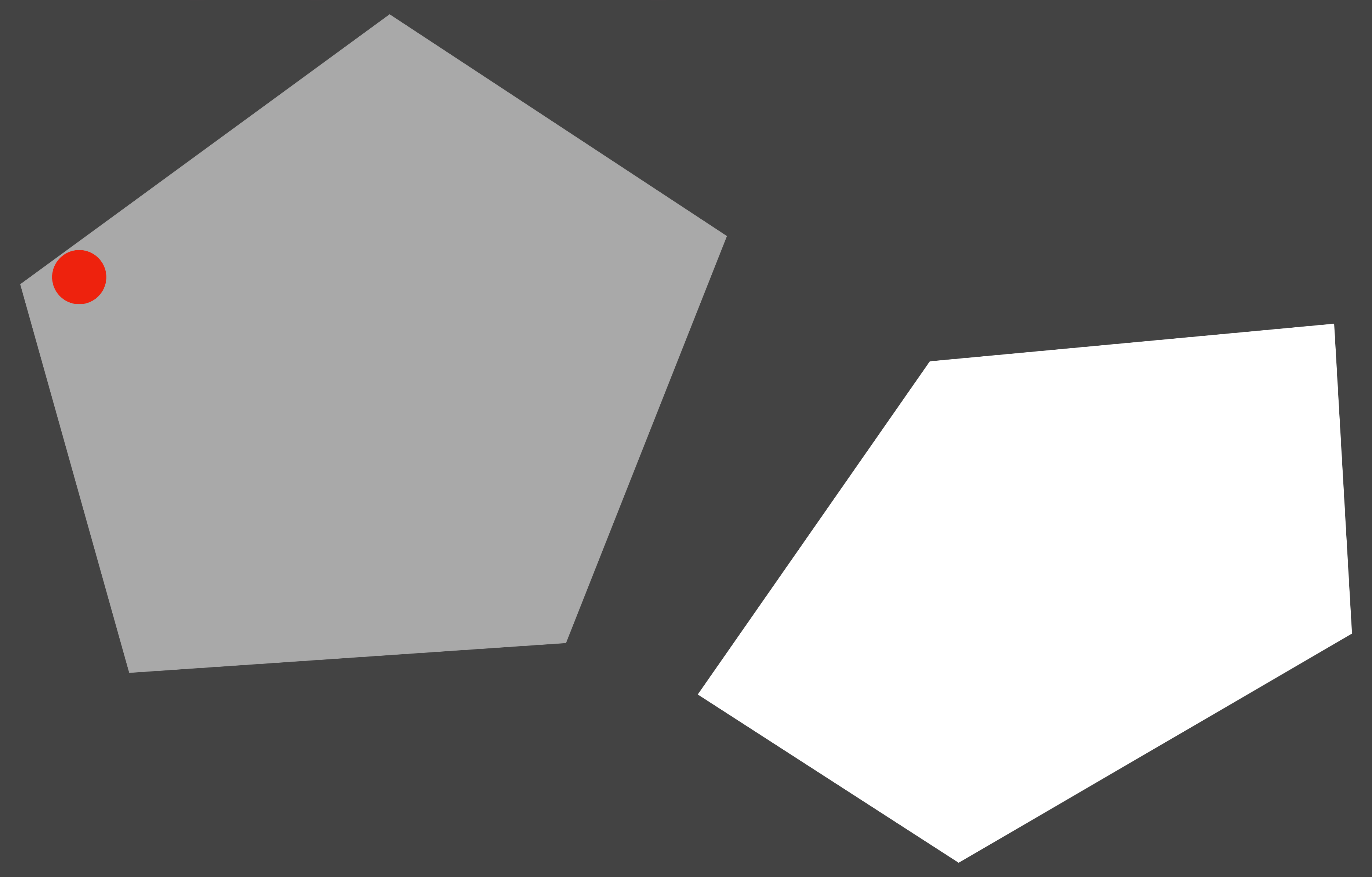

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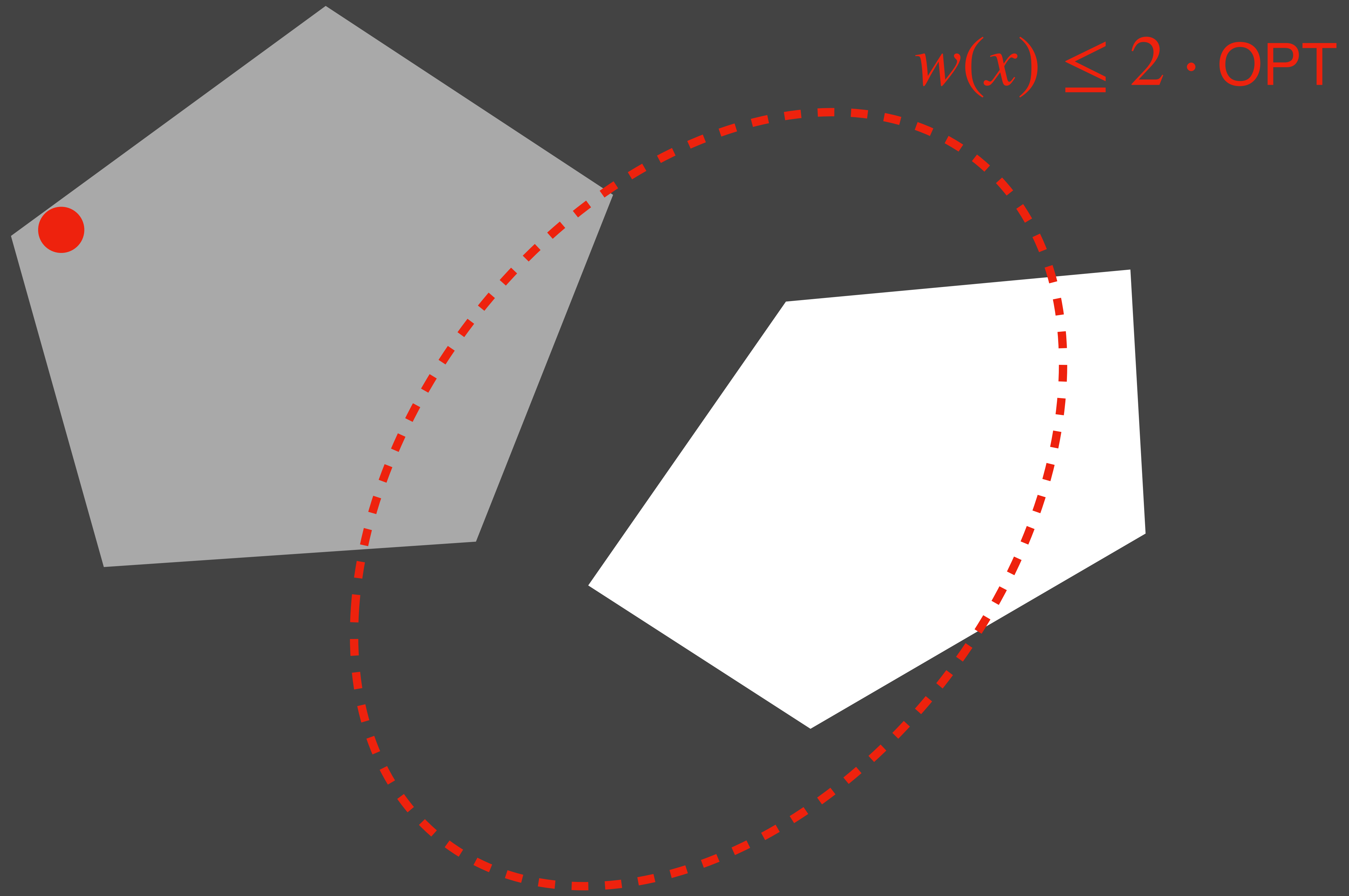

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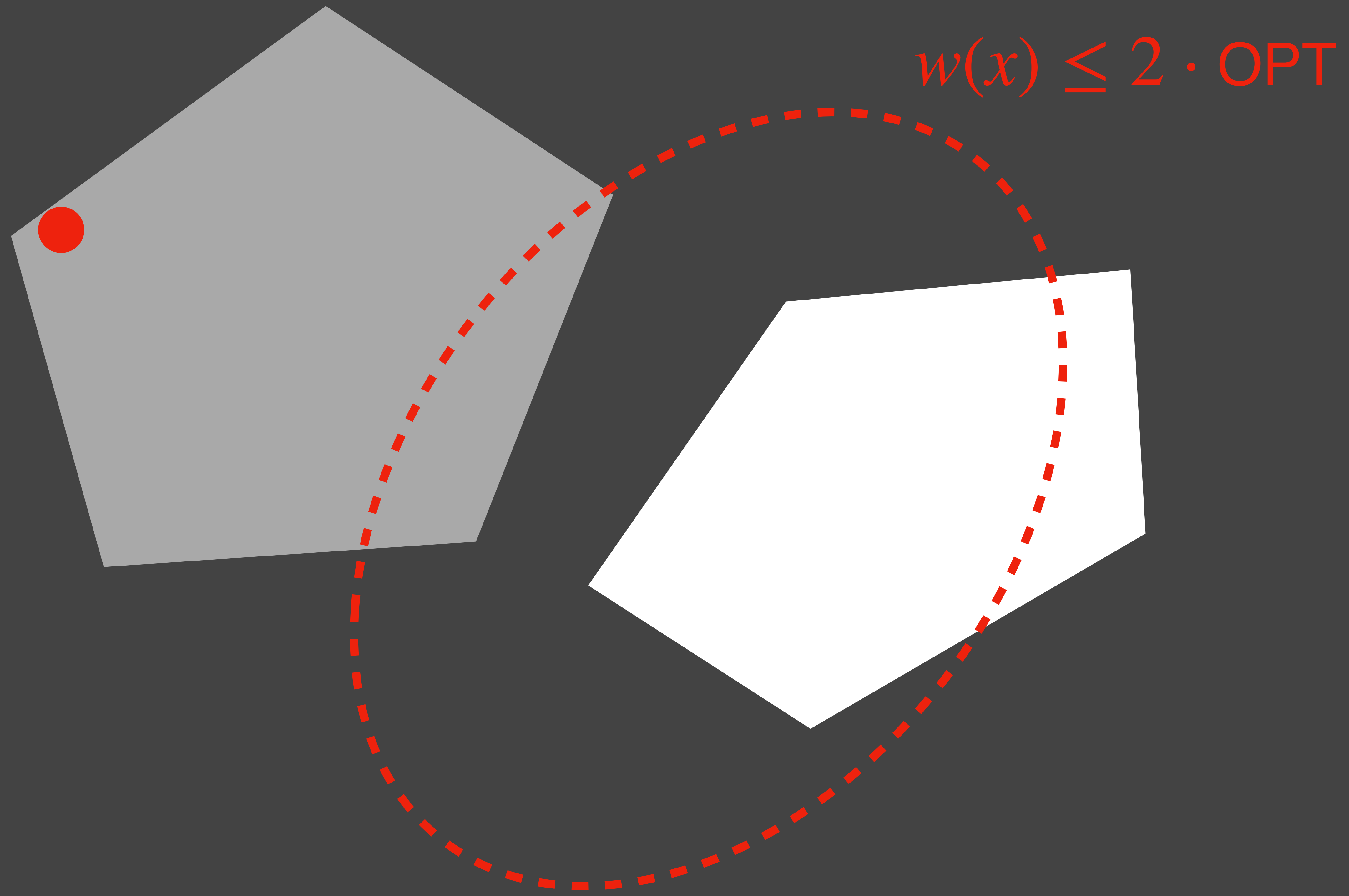

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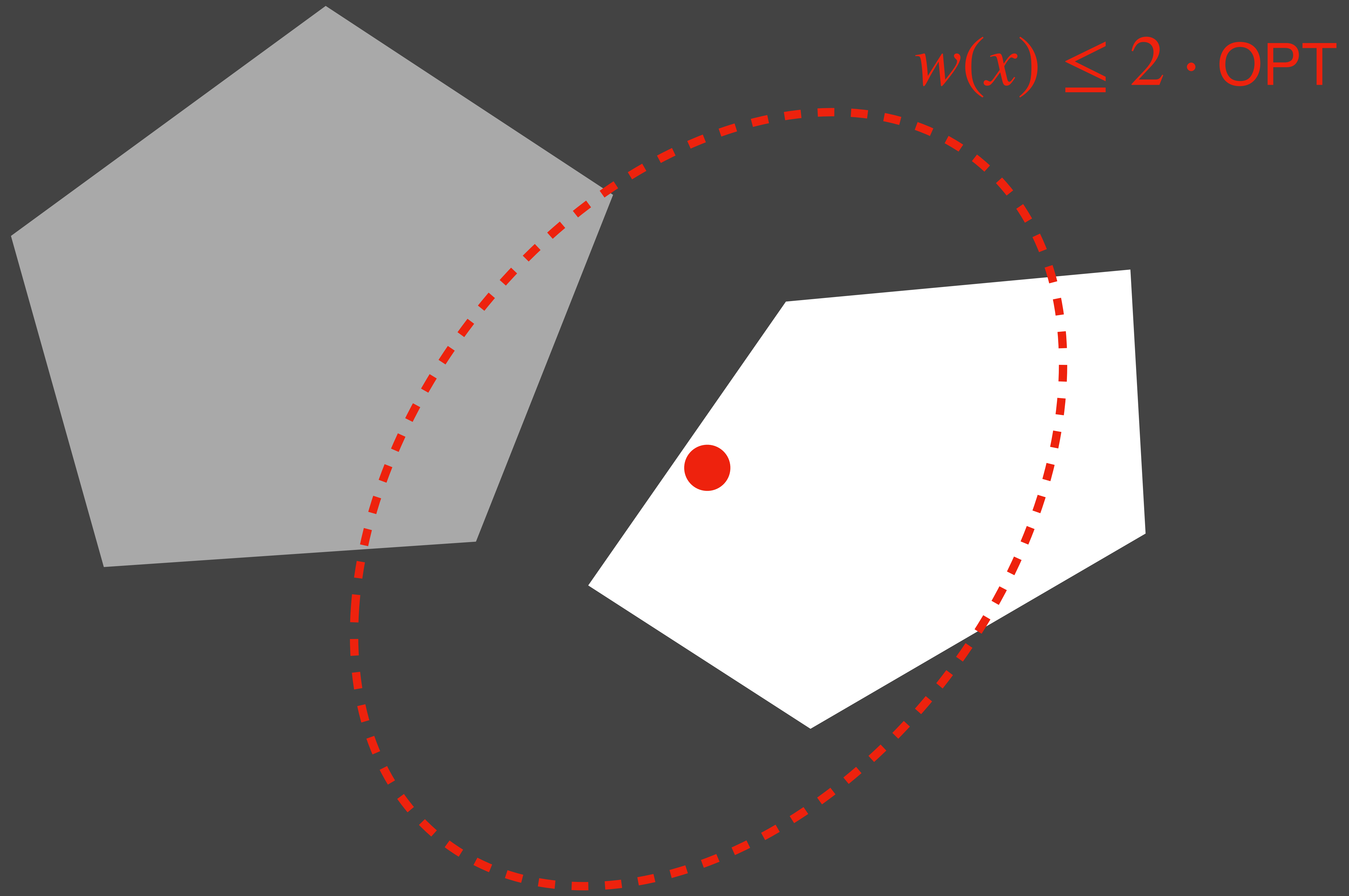

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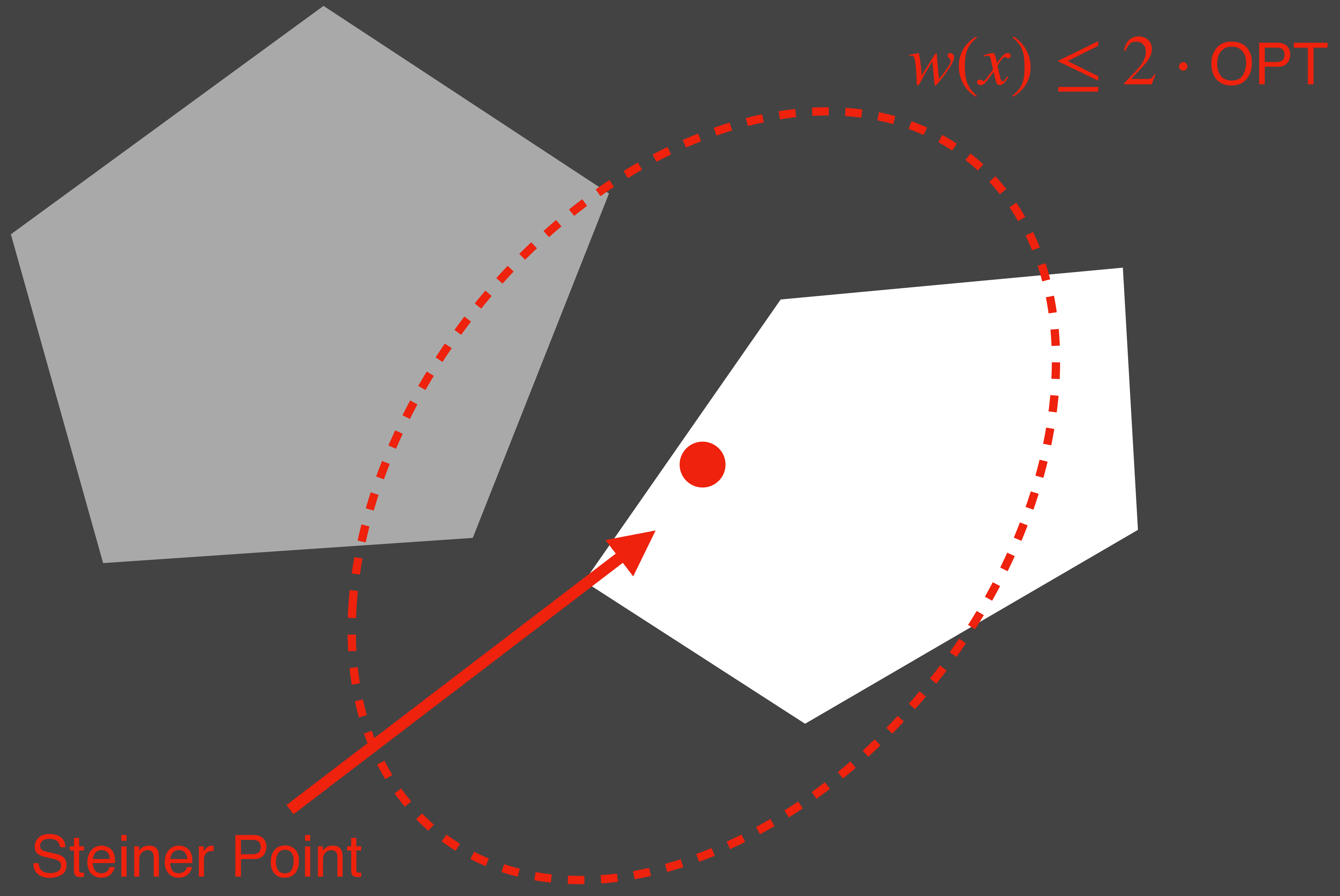

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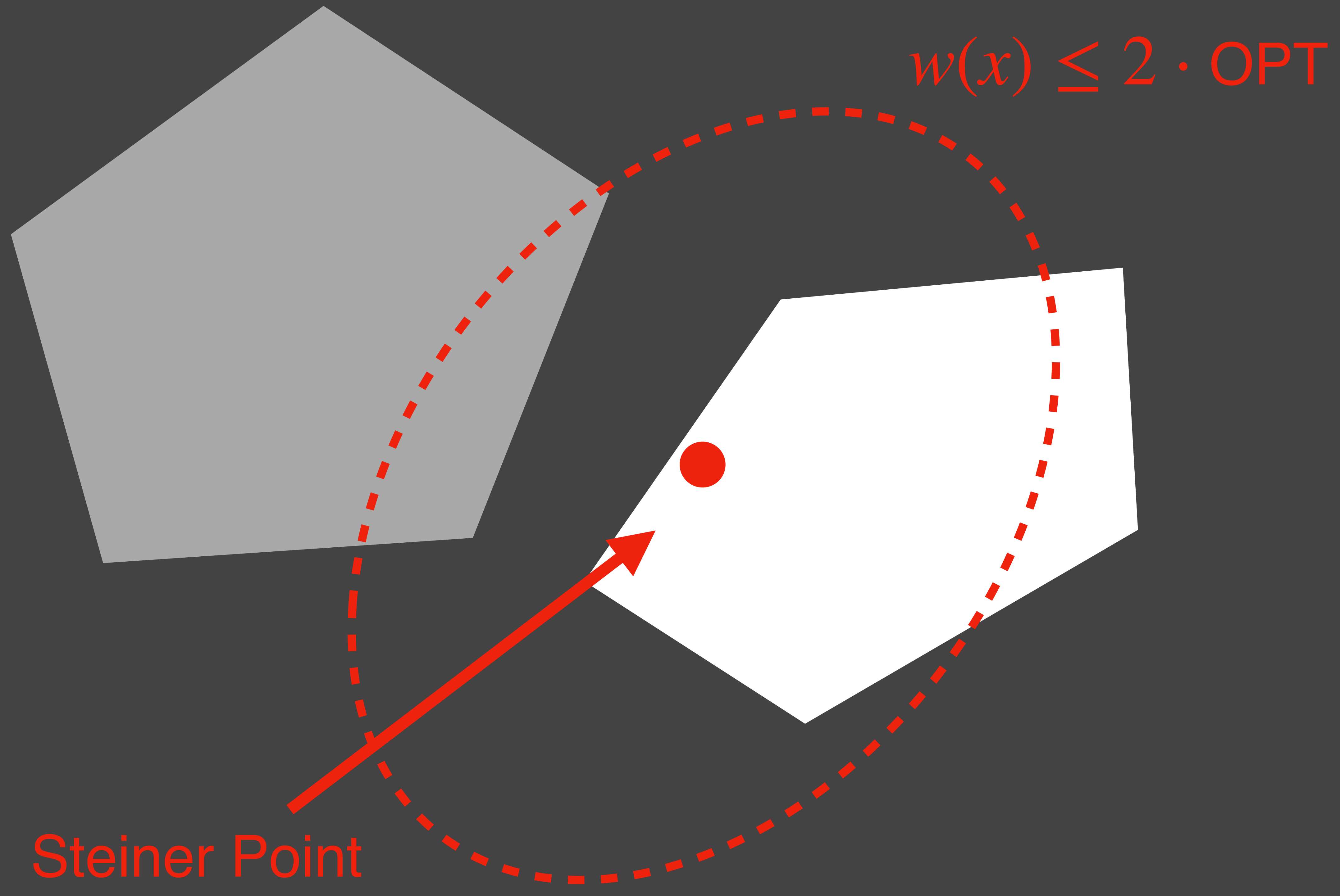

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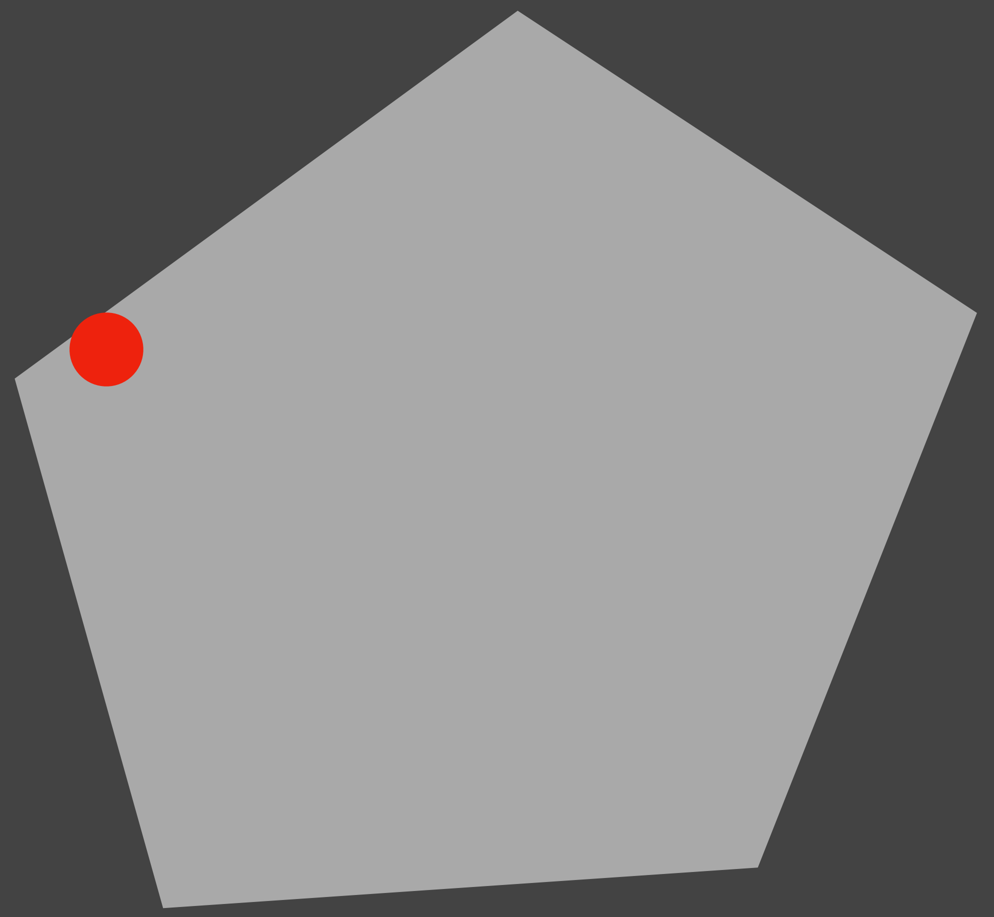

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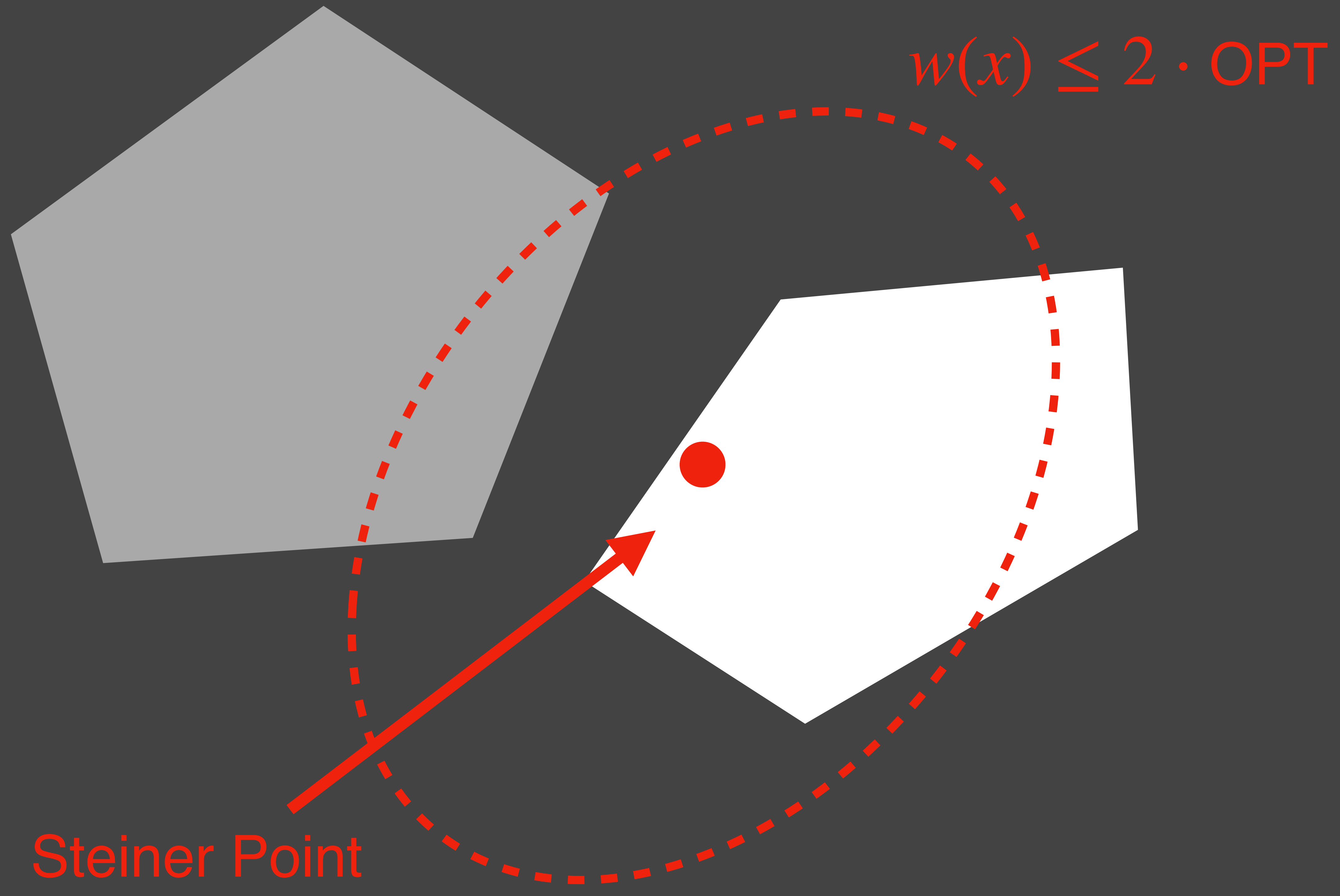

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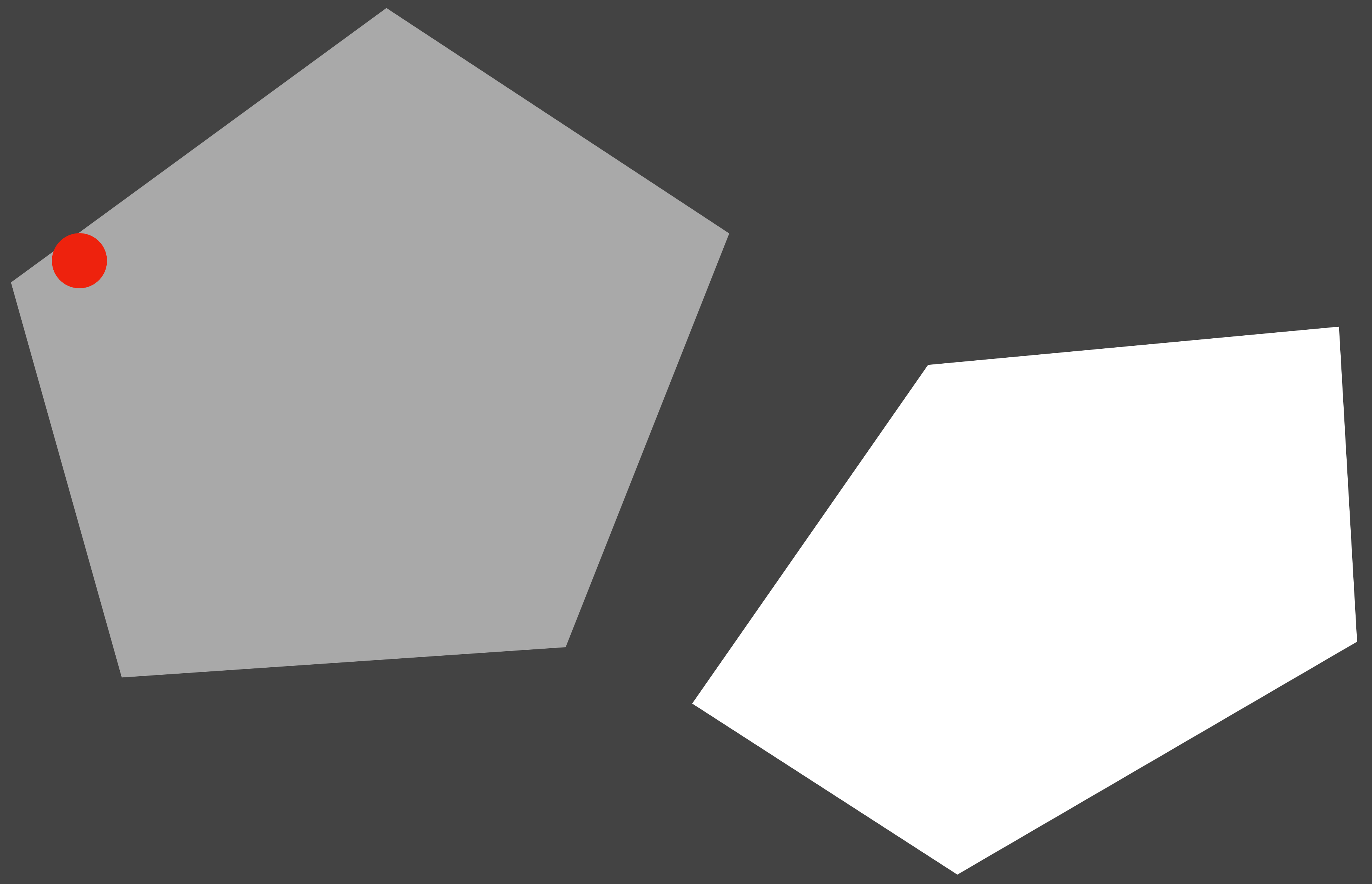

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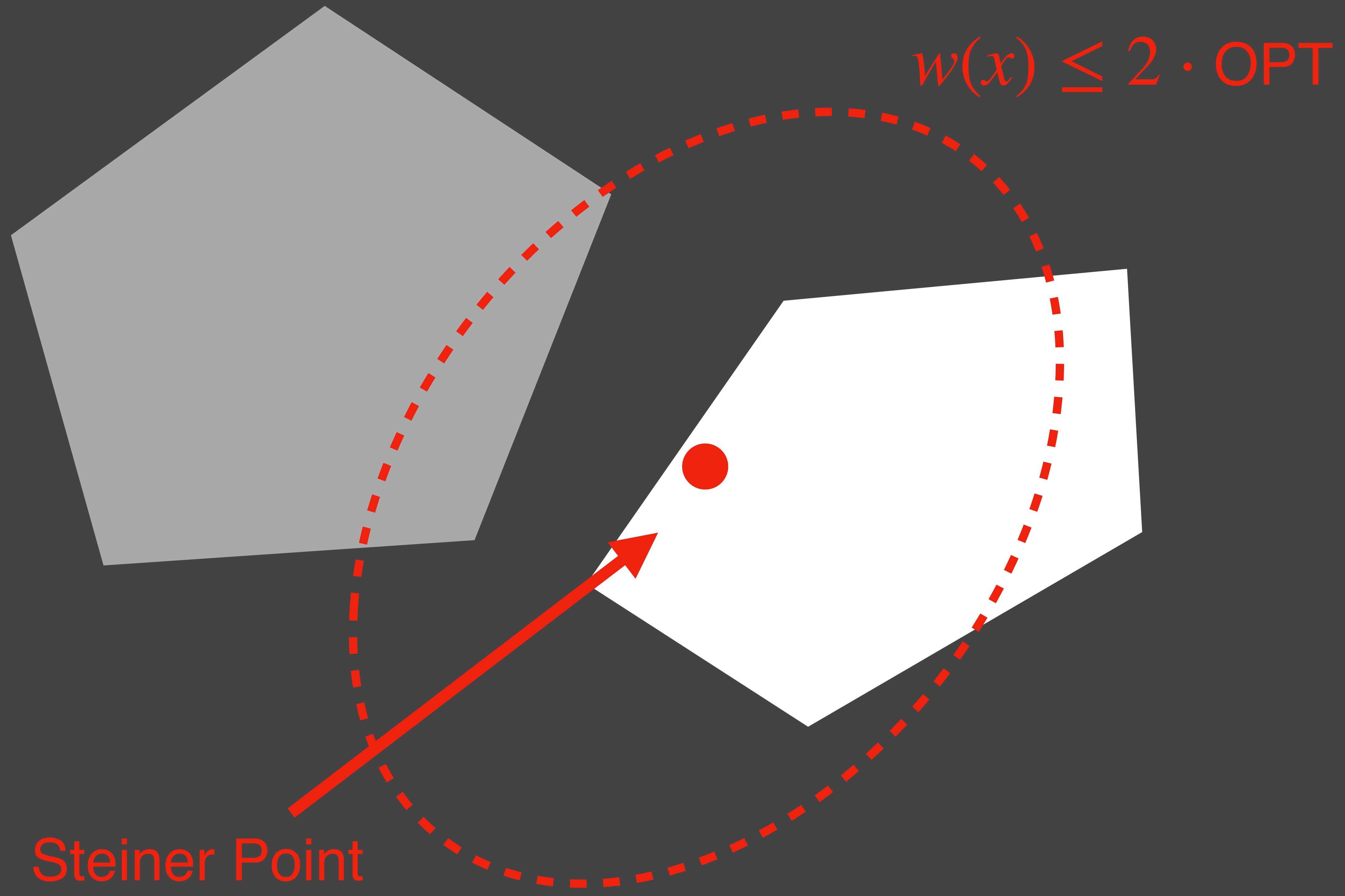

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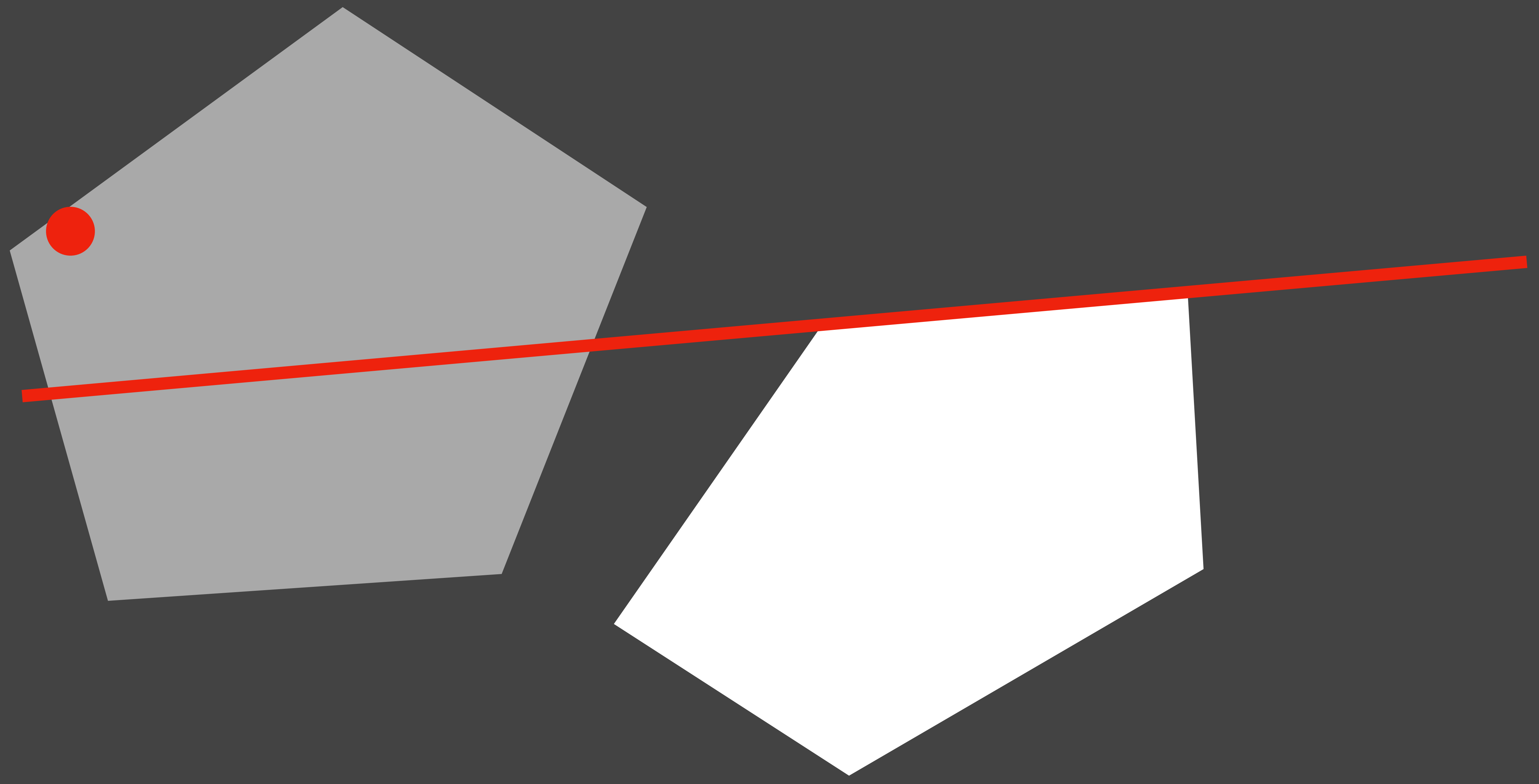

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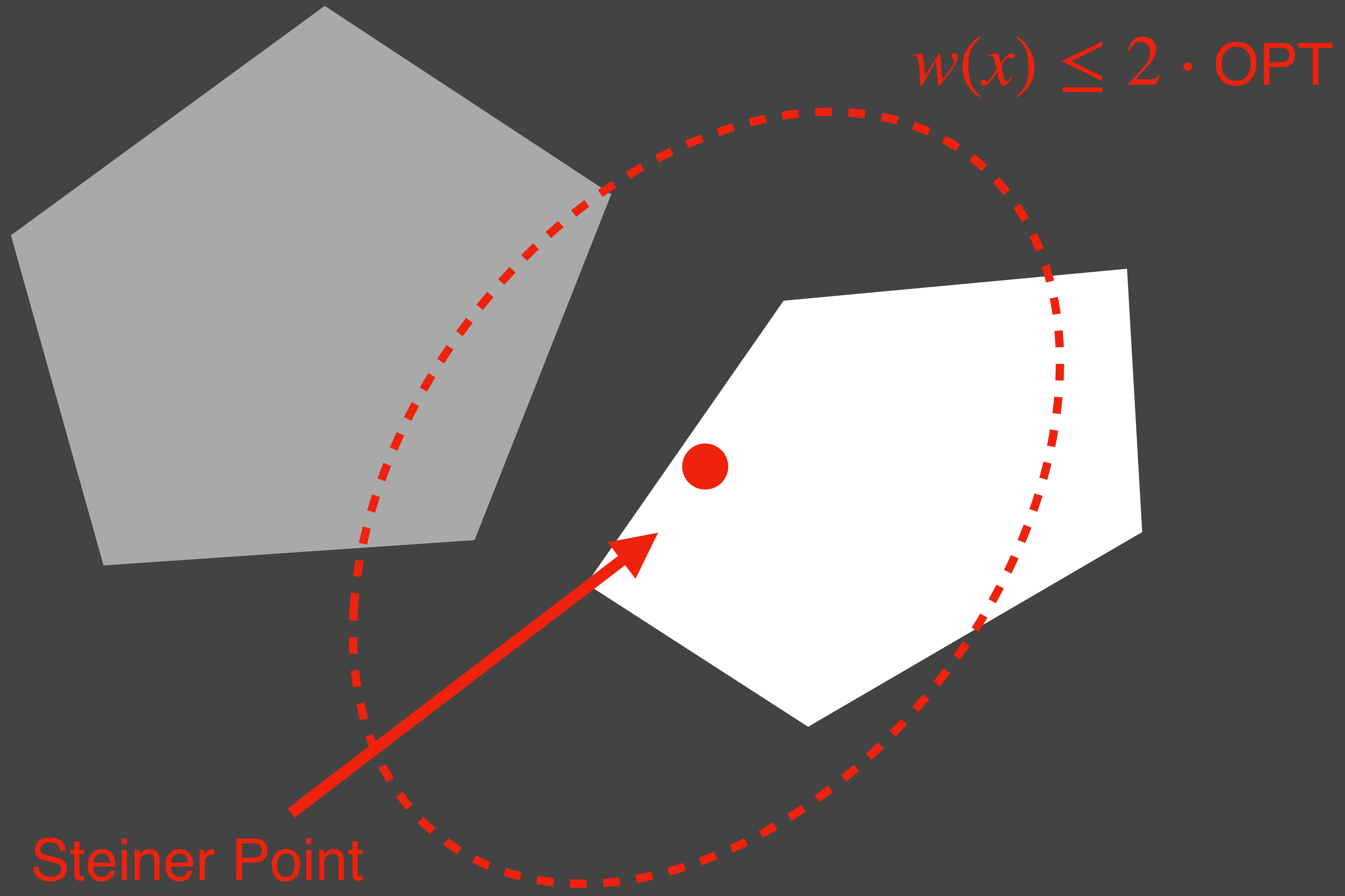

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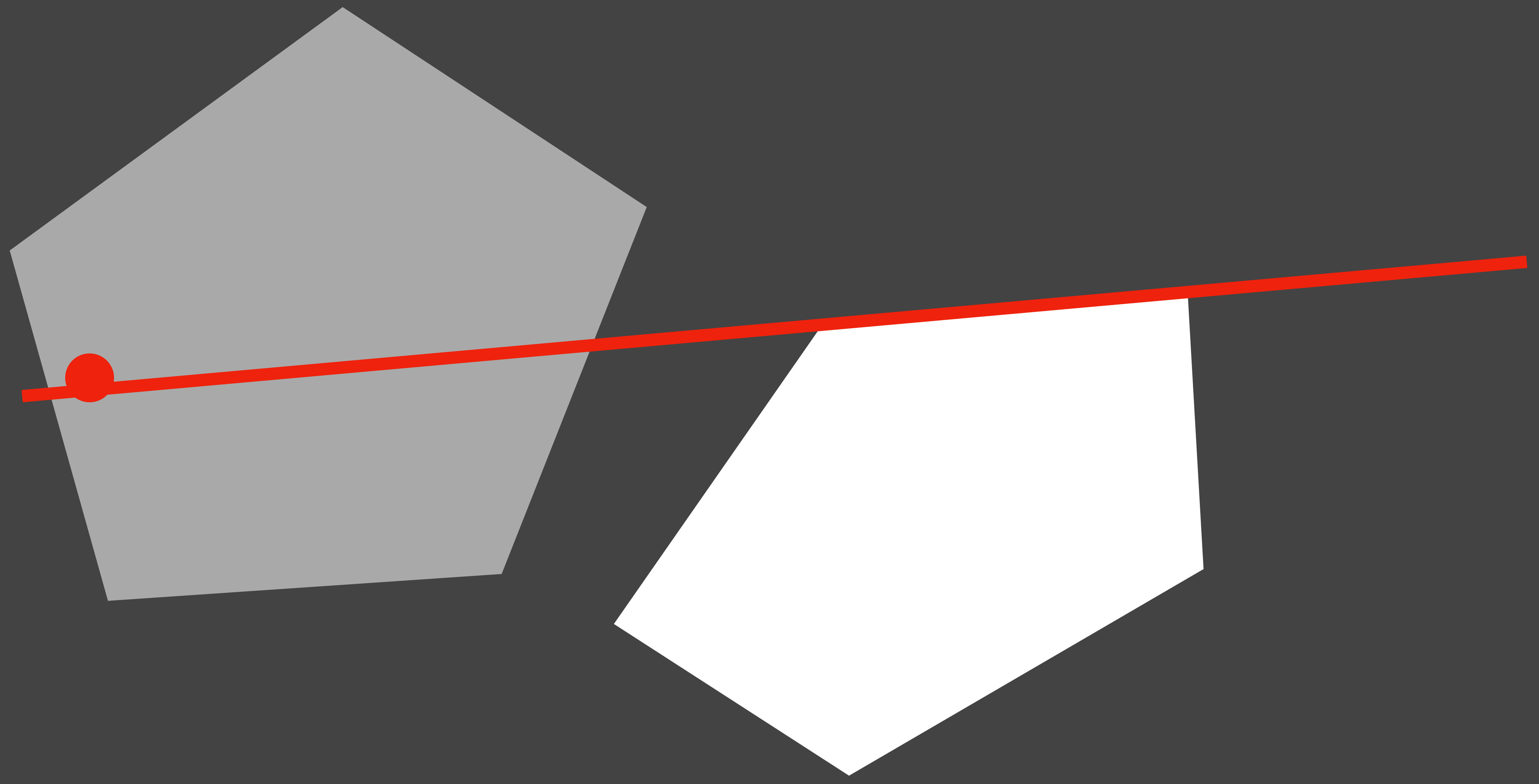

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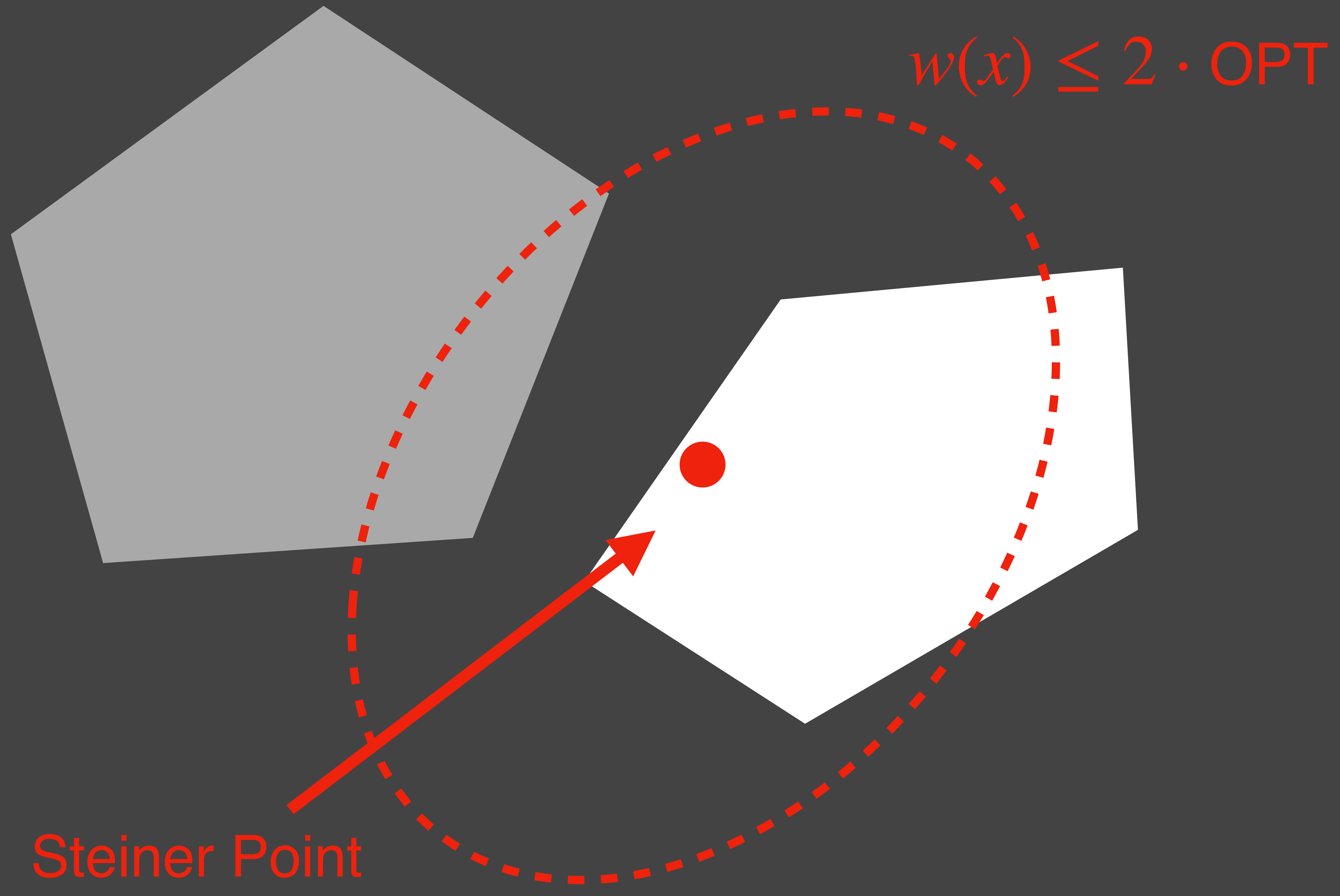

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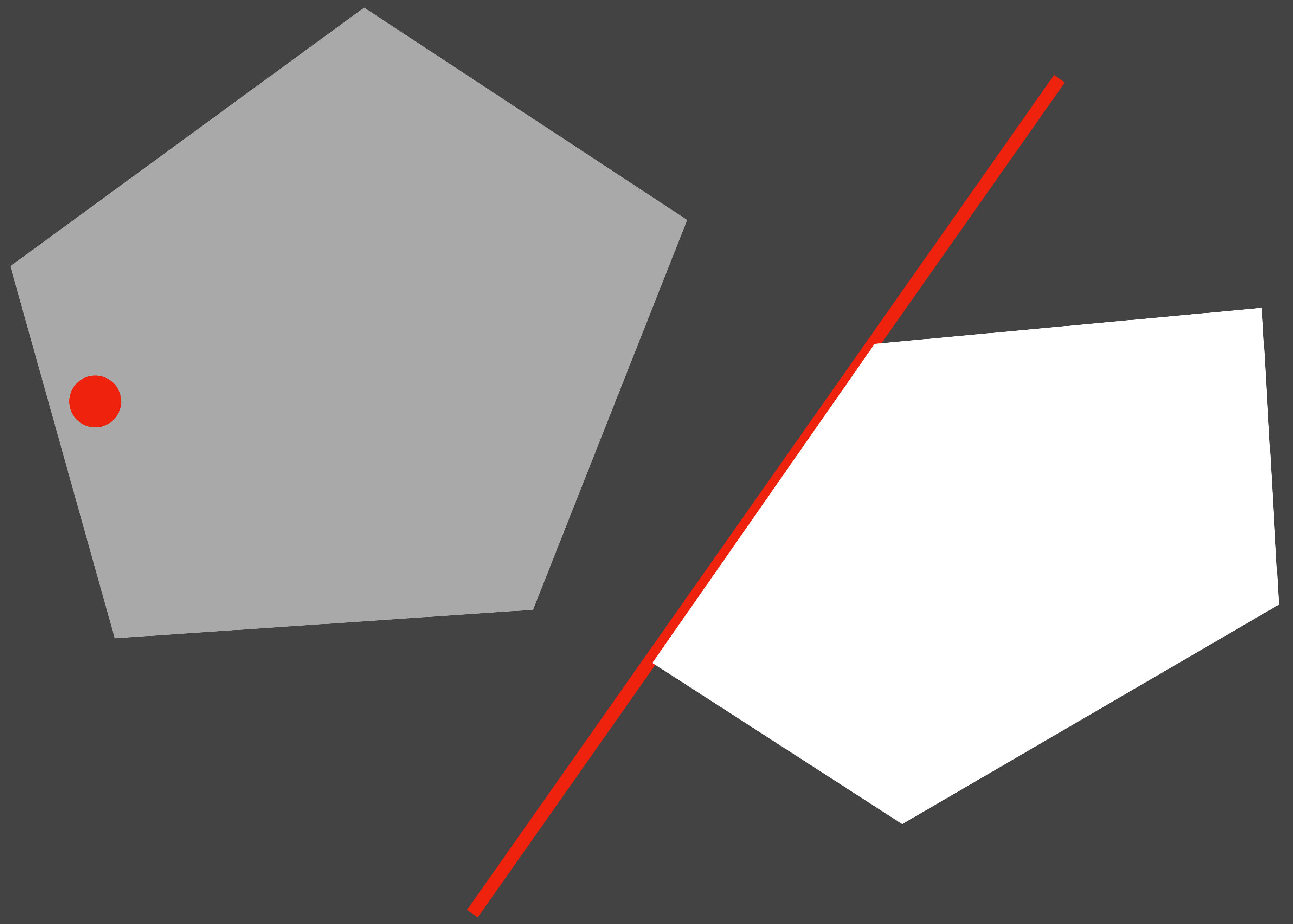

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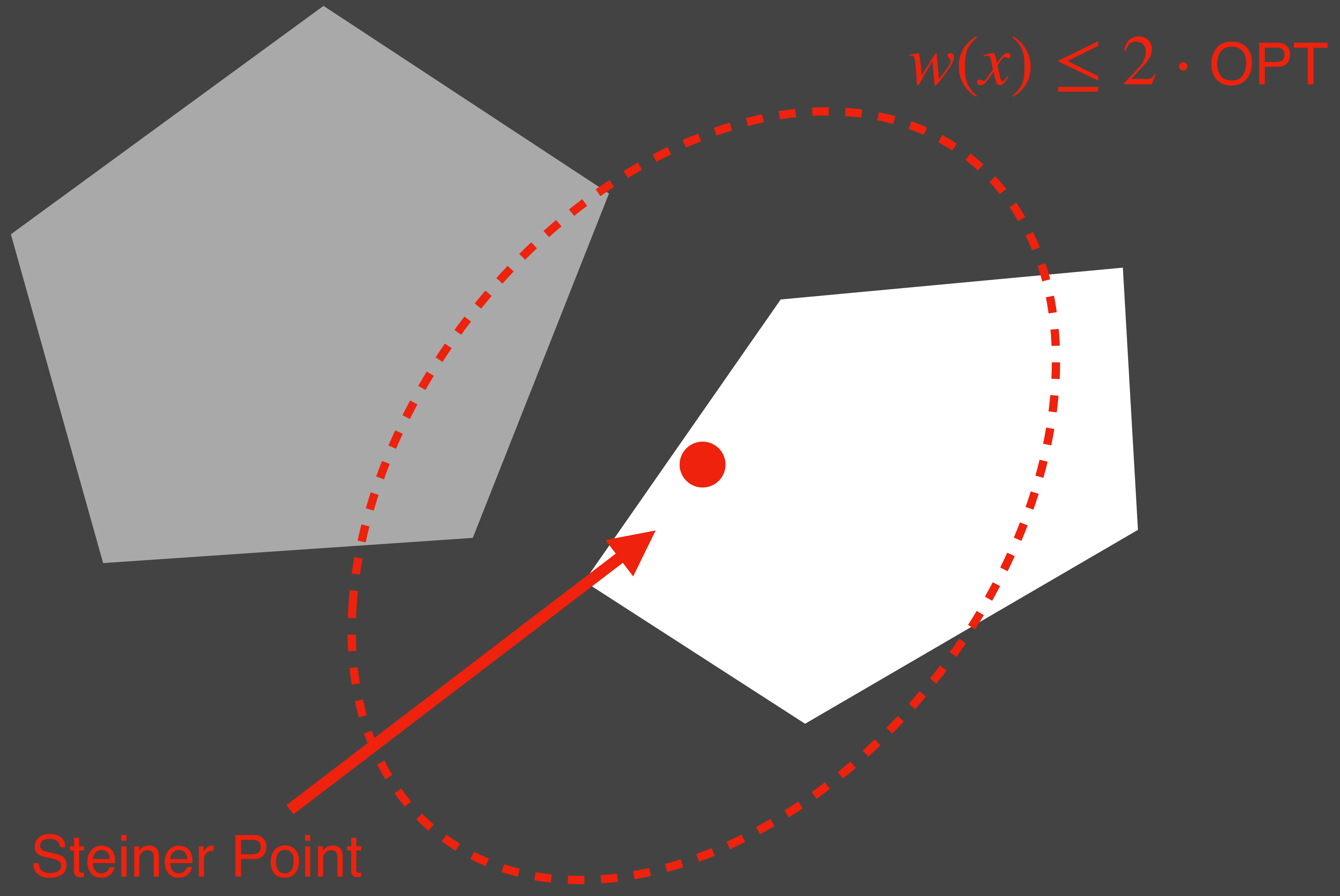

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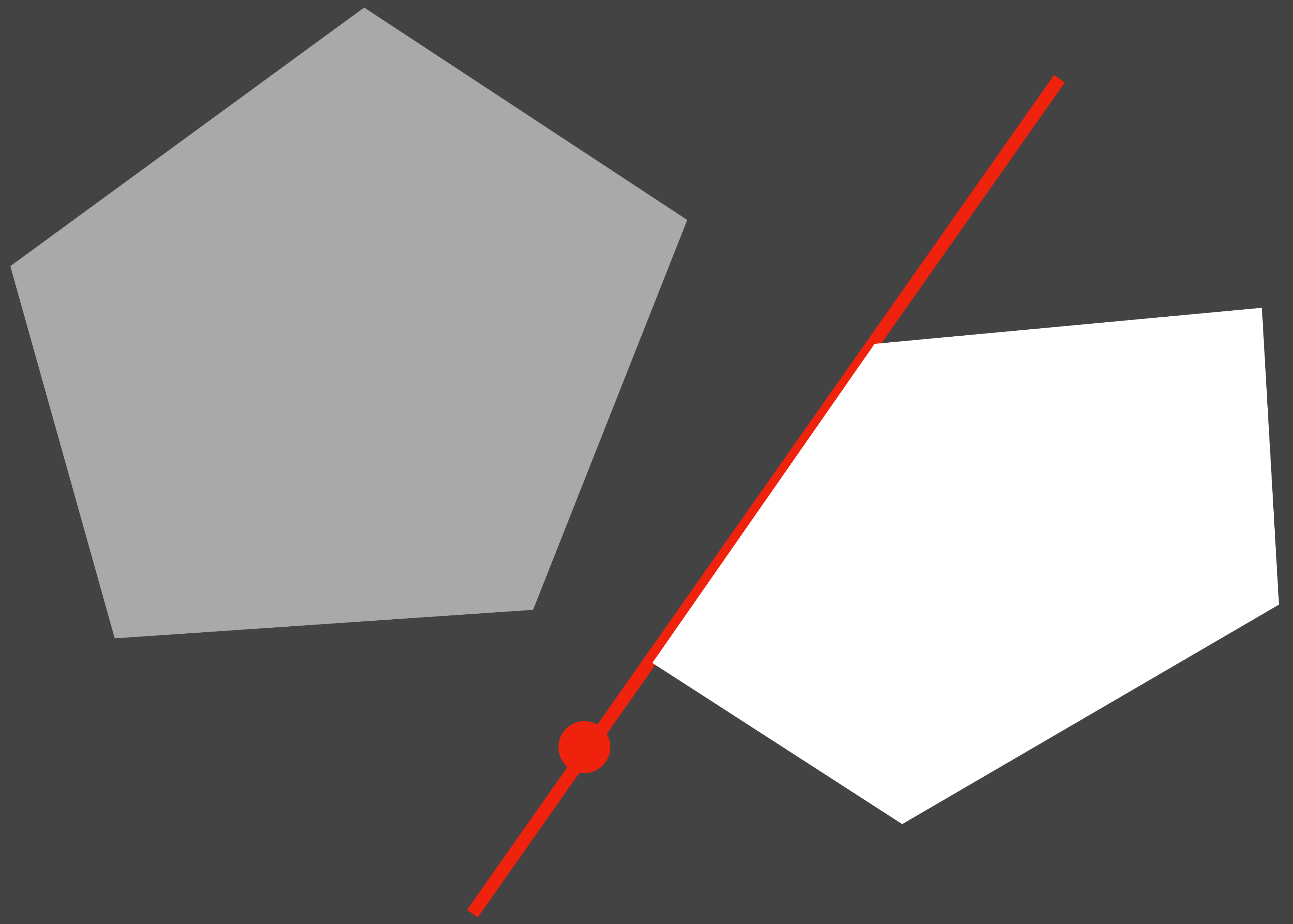

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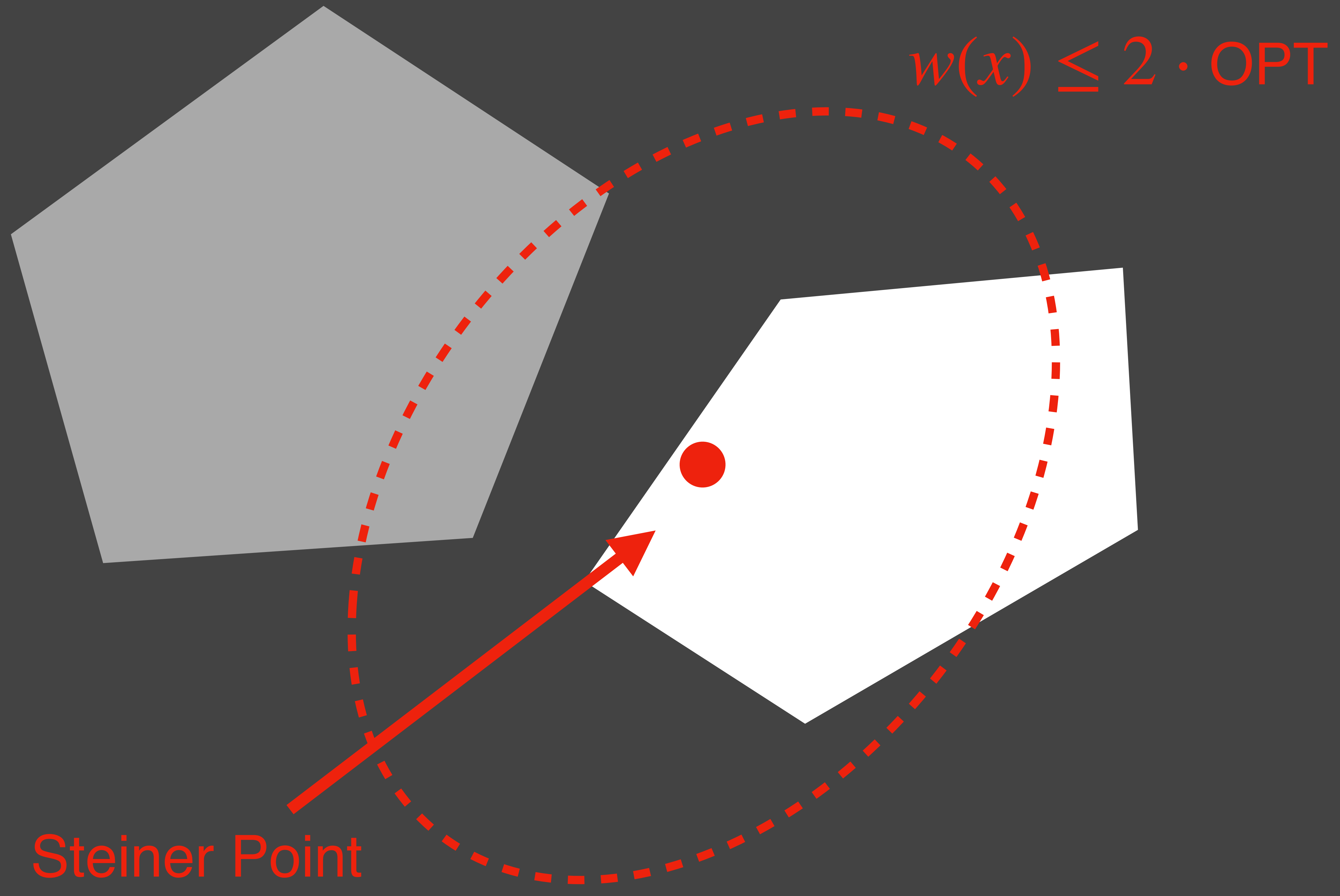

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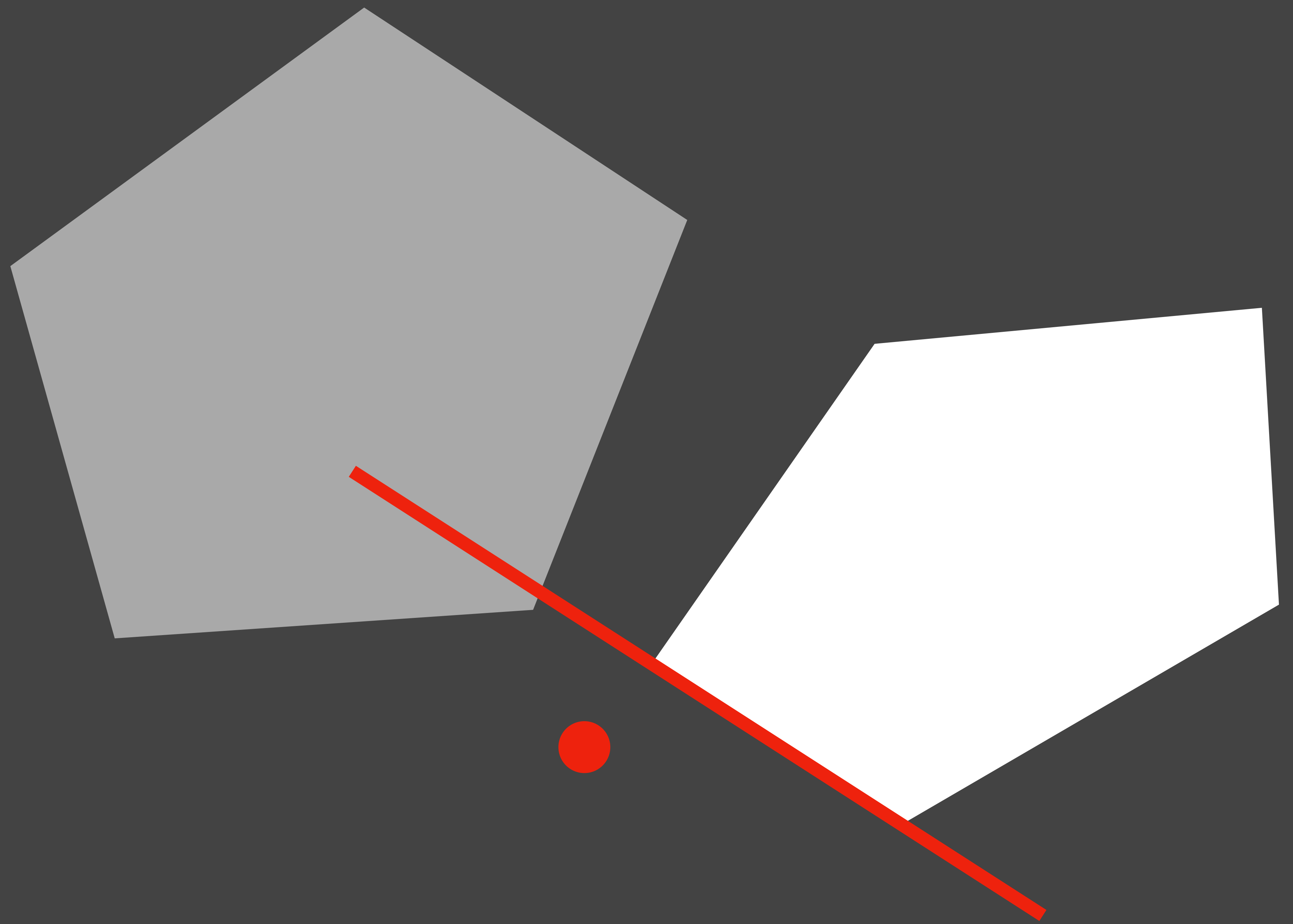

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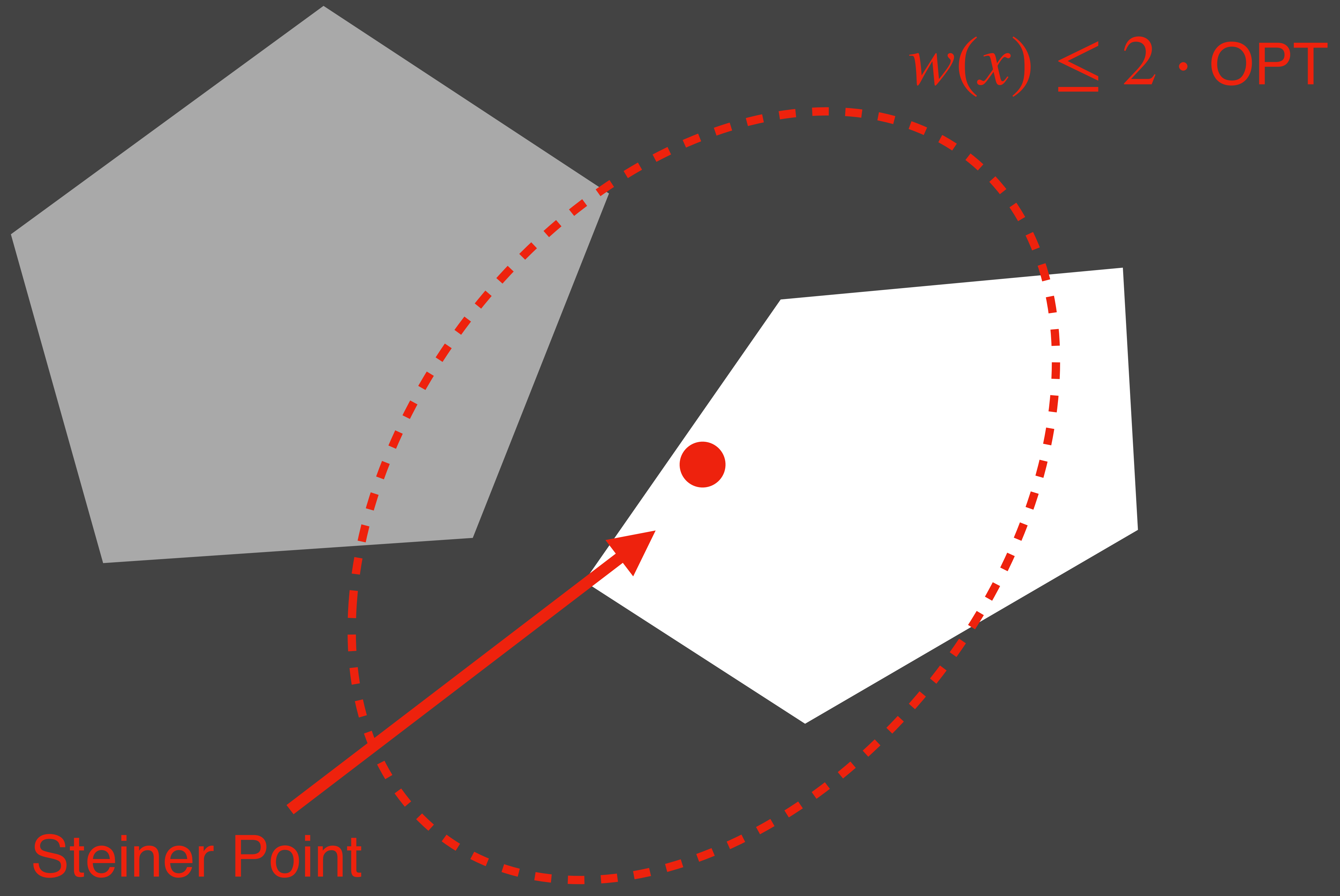

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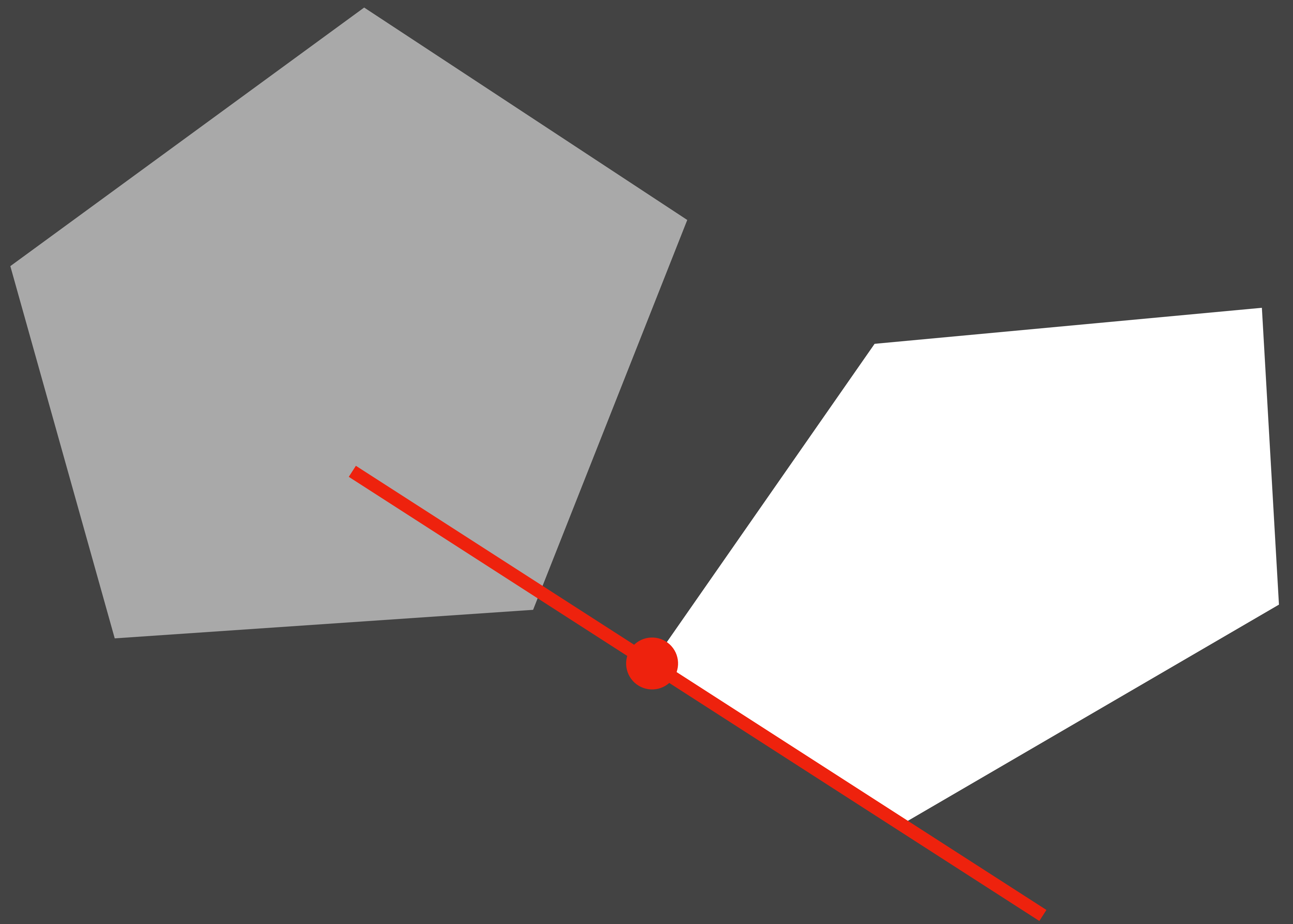

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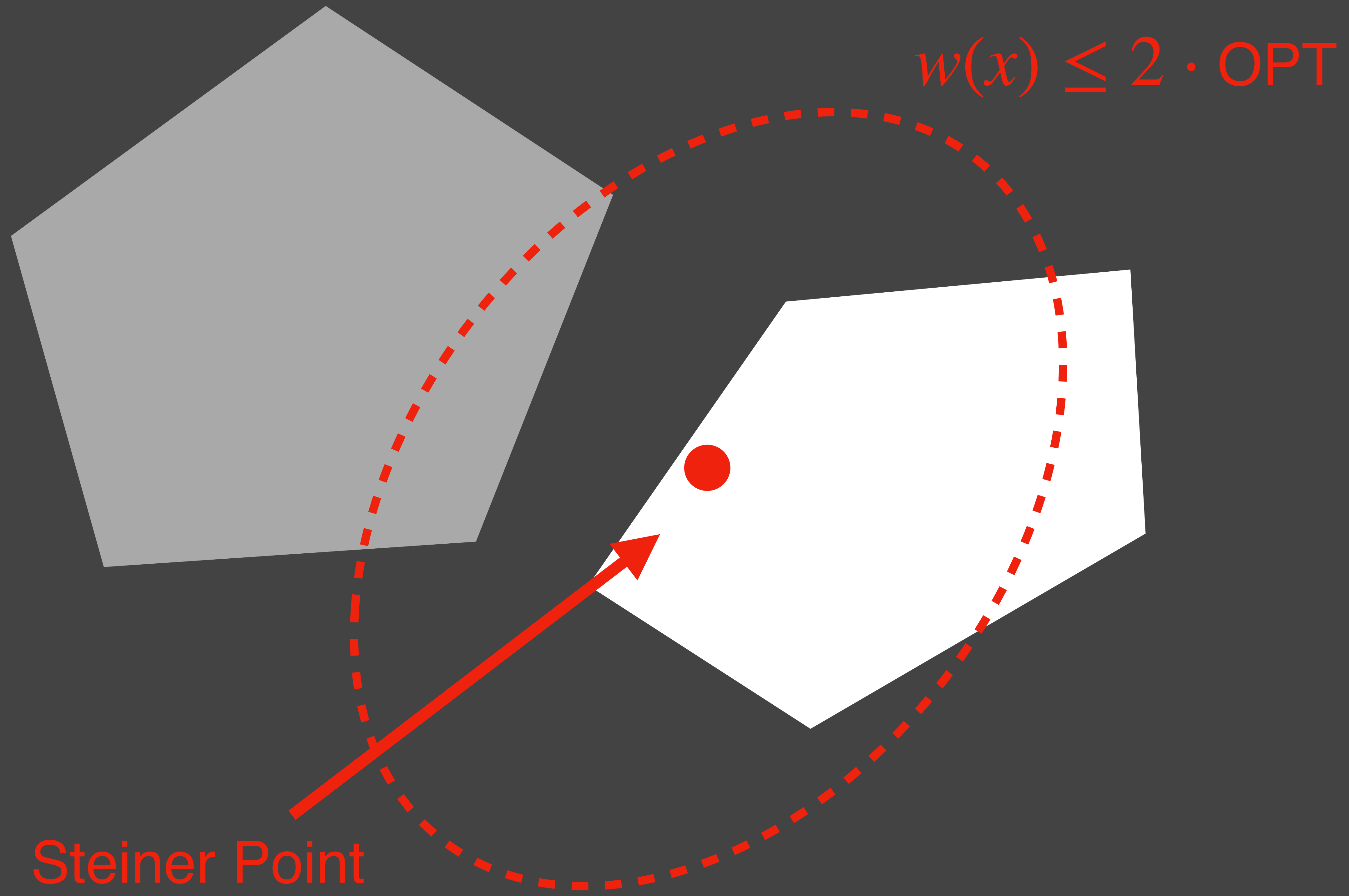

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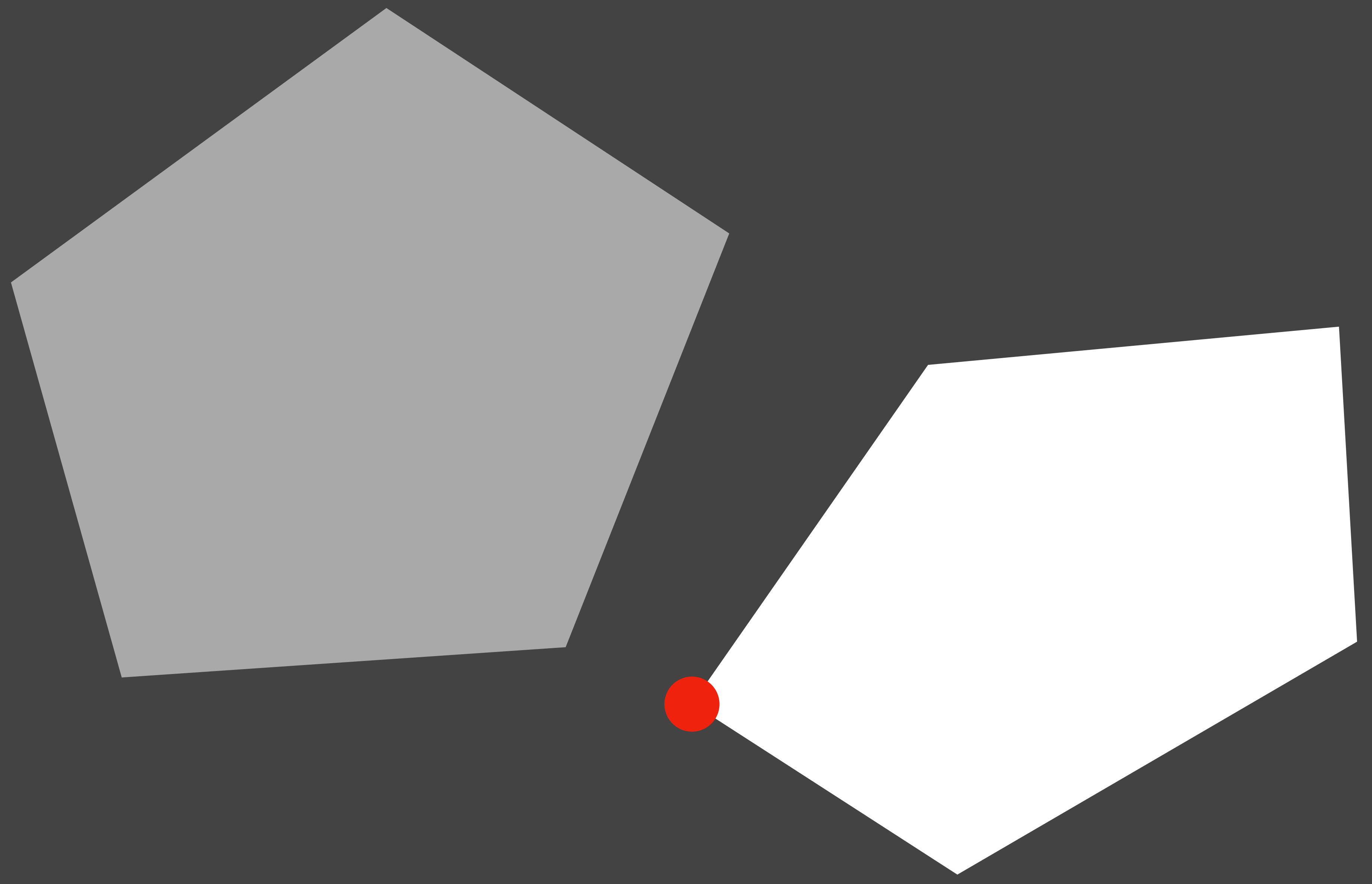

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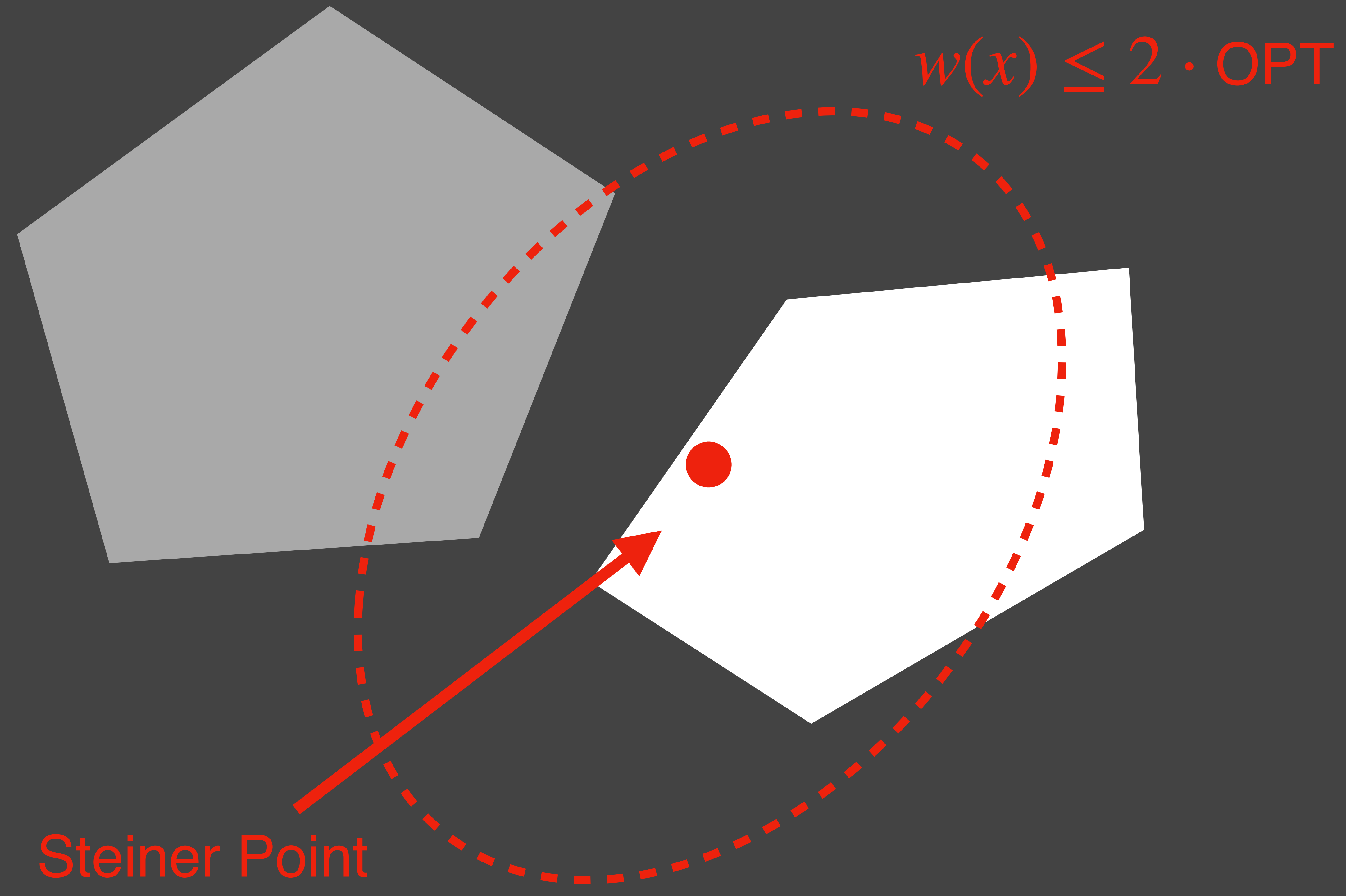
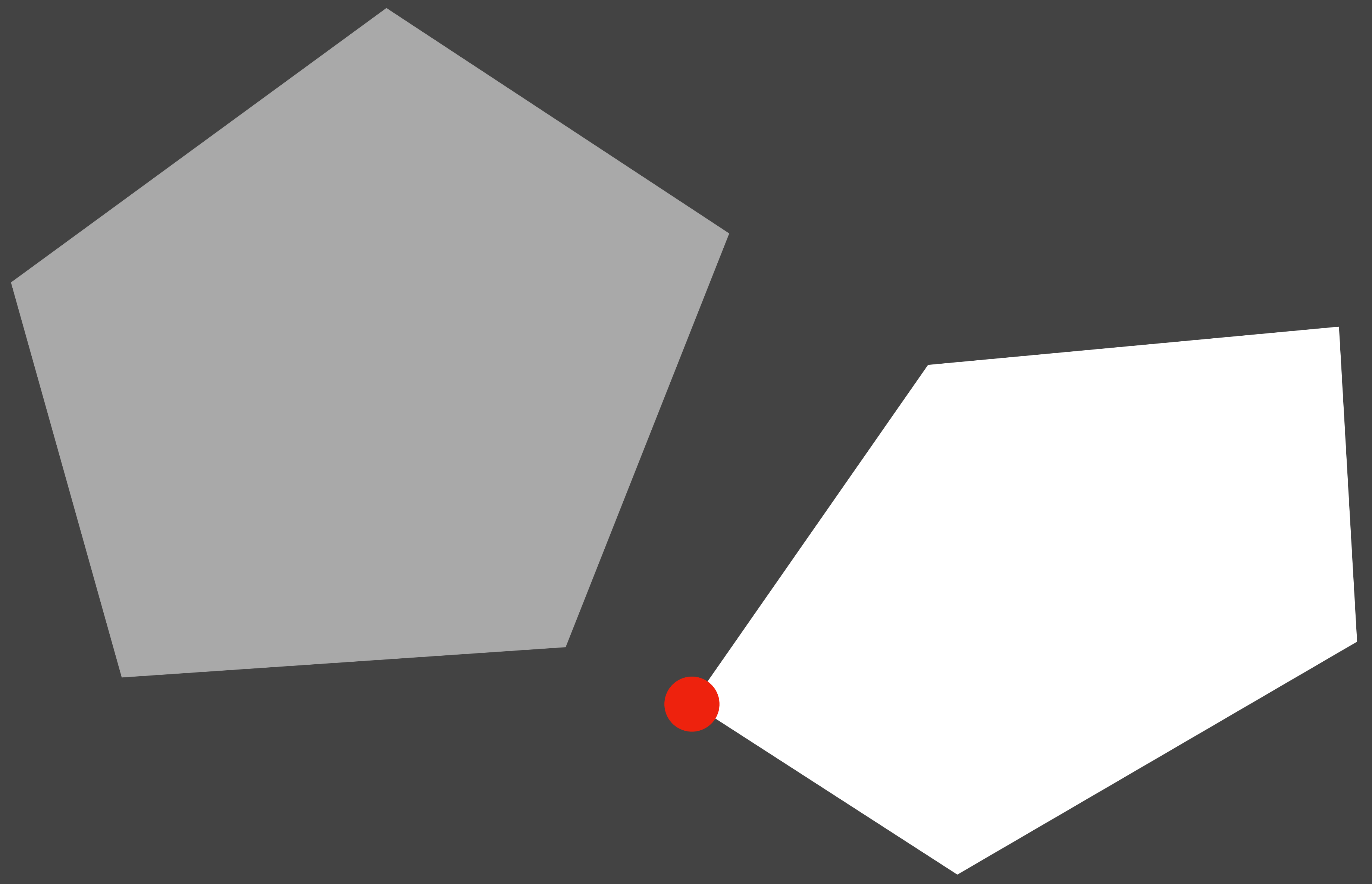
Steiner Point Algorithm for Convex Body Chasing
[AGGT 20] & [Sellke 20]



$w(x)$:= optimal cost of chasing bodies so far, then moving to x.

Our KL Projection Algorithm



Memoryless! Feature or bug?

Improved Guarantee

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Positive Body Chasing with movement $O\left(\frac{1}{\epsilon}\log\left(\frac{f}{\epsilon}\right)\right) \cdot$ OPT.

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To overcome, we go back in time and modify old duals. (ALG still online!)

Rounding

Sample Application 1 : Set Cover

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$$K_t = \left\{ x \;\middle|\; \sum_{S \ni e} x_S \geq 1 \quad \forall e \in U^t, \;\; \sum_{S} x_S \leq \beta \cdot \mathrm{OPT}^t \right\}$$

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$\text{OPT}^t :=$ cost of best cover @ time t.

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Dynamic Set Cover with:

(1) Approx $O(\log n) \cdot \beta \cdot \text{OPT}^t$.

(1) Recourse $O(\log^2 n) \cdot \text{OPT}_{\text{recourse}}(\beta)$.

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$\sum_t \Delta x$ bounded by fractional algorithm!

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Cost of solution is $O(\log n) \cdot \beta \cdot \mathsf{OPT}^t$, feasible w.h.p. @ every time t.

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Can charge every set purchase to $1/2f$ movement of fractional solution.

Sample Application 2 : Bipartite Matching

$$K_t = \left\{ x \;\middle|\; \sum_{e \in \partial(v)} x_e \leq 1 \;\; \forall v \in V, \; \sum_e x_e \geq \beta \cdot \mathsf{OPT}^t \right\}$$

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Matching recourse $= O(\epsilon^{-1}) \cdot$ [# edges updates to H] $= O(\epsilon^{-1}) \cdot [O(\log n) \cdot \mathrm{OPT}_{\text{recourse}}(\beta)]$.

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Other important + tractable families of Convex Body Chasing?

Thanks!